MA208 Quantitative Techniques for Business

Lecture 9: Probability ctd.

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Lecture 9 - Outline

Today we will talk about

- Poisson Distribution,
- Expected value of a distribution.

Poisson Distribution

The Poisson distribution is one of the most widely used probability distributions. It is usually used in scenarios where we are counting the occurrences of certain events in an interval of time or space. Here is an example of a scenario where a Poisson random variable might be used.

Example

Suppose that we are counting the number of customers who visit a certain store from 1pm to 2pm. Based on data from previous days, we know that on average $\lambda = 15$ customers visit the store. Of course, there will be more customers some days and fewer on others. Here, we may model the random variable X showing the number customers as a Poisson random variable with parameter $\lambda = 15$.

Poisson Distribution

The Poisson distribution is a discrete probability distribution that expresses the average number of events occuring in a fixed period of time or space if these events occur with a known average rate λ.

Poisson Probability Distribution

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

where λ is the average number of success per interval (also donated by μ), $e = 2.71828...$ and $x \in \{0, 1, 2, ...\}$.

Note: If we don't have a value for λ, we use $\lambda = np$.

Poisson Distribution

Properties of the Poisson distribution:

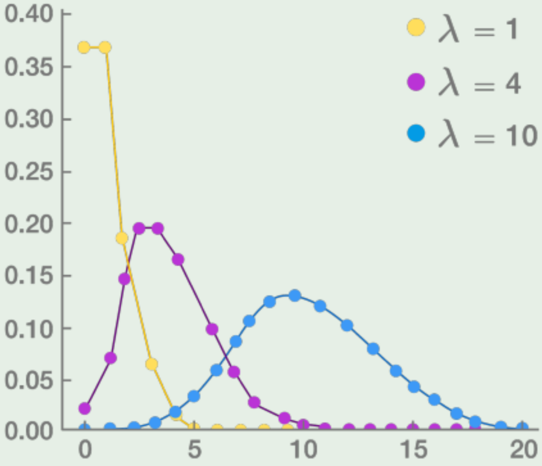
- The probability of an occurrence is the same for any two intervals of equal length.
- The occurrence or nonoccurrence in any interval is independent of the occurrence or nonoccurrence in any other interval.

The Poisson distribution is useful for modeling the number of times an event occurs in an interval of time or space, for example

- the number of patients arriving in an A&E between 11 and 12 pm,
- the number of occurrences of the DNA gene "ACGT" in a gene.

Poisson Distribution

Example



Poisson Distribution

Example

Given that a bank receives on average six bad checks per day, what is the probability that it will receive four in a day?

Solution

$$\lambda = 6$$

$$P(X=4) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \frac{e^{-6}\,6^{4}}{4!} \approx 0.132$$

Poisson Distribution

Example

A life insurance salesman sells on average three life insurance policies per week. Use the Poisson distribution formula to calculate the probability that in a given week he will sell

(a) some policies,

(b) two or more policies, but less than five policies.

Solution

$\lambda = 3.$

(a) $P(X>0) = 1 - P(X=0)$

$= 1 - \frac{e^{-3} \cdot 3^0}{0!} = 1 - 4.9787 \times 10^{-10} = 0.95021$

(b) $P(2 \le X < 5) = P(X=2) + P(X=3) + P(X=4)$

$= \frac{e^{-3} \cdot 3^2}{2!} + \frac{e^{-3} \cdot 3^3}{3!} + \frac{e^{-3} \cdot 3^4}{4!}$

$= 0.61611$

Expected Value

If you have a collection of numbers $a_1, a_2, ..., a_n$, their average is a single number that describes the whole collection. Now, consider a random variable X. We would like to define its average, or as it is called in probability, its expected value or mean. The expected value is defined as the weighted average of the values in the range.

Expected Value

Suspose a random variable X can take

value	x_1	with probability	p_1,
"	x_2	"	p_2,
"	x_3	"	p_3,
	...		
"	x_k	"	p_k.

Then the expected value of the random variable X is defined as

Expected Value

$$E(X) = \sum_{i=1}^{k} x_i p_i$$

Note: The expected value $E(X)$ is also called the mean of the probability distribution and often denoted by μ.

Expected Value

Consider the following example from Lecture 8:

Example

Two balls are drawn in random in a succession without replacement from an urn containing four red balls and six black balls. We found the possibilities of all possible outcomes.

1. What is the expected number of red balls?
2. If we perform the experiment 1000 times, how many red balls can we expect to get?

Below are the values we got in Lecture 8.

Possible outcomes	RR	RB	BR	BB
x_i	2	1	1	0
$P(x_i)$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{4}{15}$	$\frac{1}{3}$

Expected Value

Solution

1.

$$
\begin{aligned}
E(X) &= \sum_{i=1}^{k} x_i p_i \\
&= 2 \cdot \frac{2}{15} + 1 \cdot \frac{4}{15} + 1 \cdot \frac{4}{15} + 0 \cdot \frac{1}{3} \\
&= \frac{4}{5} = 0.8
\end{aligned}
$$

2.

$$1000 \times E(X) = 1000 \times 0.8 = 800$$

If we perform the experiment 1000 times, we would expect to get 800 red balls.

Expected Value

The concept of expected value of a random variable is one of the most important concepts in probability theory. It was first devised in the 17th century to analyze gambling games and answer questions such as:

- How much do I gain - or lose - on average, if I repeatedly play a given gambling game?
- How much can I expect to gain - or lose - by performing a certain bet?

Example

If you play a game where you gain €2 with probability 1/2 and you lose €1 with probability 1/2, then the expected value of the game is 50 cent:

$$E(X) = (\frac{1}{2})(+€2) + (\frac{1}{2})(-€1) = €0.50$$

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Expected Value

Example

You are invited to play a game in which five fair coins are tossed in a fair way. Denote by X the number of heads that the five coins show together.

(a) What are the chances that (i) $X = 0$, (ii) $X = 1$, (iii) $X = 2$, (iv) $X = 3$, (v) $X = 4$, (v) $X = 5$?

(b) You win €$X + X^2$ if an even number of the coins show heads, and you lose €$X + X^2$ if an odd number of them show heads. What are your expected winnings in this game?

Answer

(a) Binomial distribution, so $P(x=0) = \binom{5}{0}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0 = \frac{1}{32}$,
$P(x=1) = \frac{5}{32}$, ...

(b) Expected winnings $= \sum P(X=x) \cdot$ winnings
$= \left(\frac{1}{3}\right) \cdot (0+0)^2 - \frac{5}{32}(1+1)^2 + \ldots$ Finish yourself!