MA208 Quantitative Techniques for Business

Lecture 7: Probability ctd.

Dr Kirsten Pfeiffer

School of Mathematics, Applied Mathematics and Statistics
NUI Galway

Lecture 7 - Outline

Today we will talk about

- Independent and dependent events,
- Mutually exclusive (disjoint) events,
- De Morgan's Laws,
- Finding probabilities.

Independent and Dependent Events

If the occurence or non-occurence of E_1 does not effect the probability of occurence of E_2, then

$$P(E_2|E_1) = P(E_2)$$

and E_1 and E_2 are said to be independent events. Otherwise they are said to be dependent events. This has an effect on the probability that both events occur:

- If E_1 and E_2 are **dependent** events, then
$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2|E_1)$$
- If E_1 and E_2 are **independent** events, then
$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

Example

A fair die is tossed twice. Find the probability of getting a 4 or 5 on the first toss and a 1, 2 or 3 on the second toss.

Solution

$$P(E_1) = P(\text{"4 or 5"}) = \frac{2}{6} = \frac{1}{3}$$

$$P(E_2) = P(\text{"1, 2, 3"}) = \frac{3}{6} = \frac{1}{2}$$

E_1 and E_2 are independent events, so

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

$$= \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

Independent and Dependent Events

Example

Two balls are drawn successively without replacement from a box which contains 4 white balls and 3 red balls. Find the probability that

(a) the first ball drawn is white and the second is red,

(b) both balls are red.

Solution

(a) The second event is dependent on the first.

$P(E_1) = P(\text{"white"}) = \frac{4}{7}$

There are 6 balls left. Out of those, three are red.

So $P(E_2 | E_1) = P(\text{"red | white"}) = \frac{3}{6} = \frac{1}{2}$

Dependent events, so

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2 | E_1) = \frac{4}{7} \cdot \frac{1}{2} = \frac{2}{7}$$

Solution

(b) Also dependent events.

$P(R_1) = \frac{3}{7}$

$P(R_2 \mid R_1) = \frac{2}{6} = \frac{1}{3}$

$$P(R_1 \cap R_2) = P(R_1) \cdot P(R_2 \mid R_1)$$
$$= \frac{3}{7} \cdot \frac{1}{3}$$
$$= \frac{1}{7}$$

Independent and Dependent Events

Example

Suppose that the probability of being killed in a single flight is $P_k = \frac{1}{4 \times 10^6}$ based on available statistics. Assume that different flights are independent. If a businesswoman takes 20 flights per year, what is the probability that she is killed in a plane crash within the next 20 years? (Let's assume that she will not die because of another reason within the next 20 years.)

Solution

The total number of flights she will take in 20 years is

$$N = 20 \times 20 = 400$$

Let P_s be the probability that she survives a given single flight. Then

$$P_s = 1 - P_k$$

Solution

The flights are independent, so the probability that she will survive all $N = 400$ flights is

$$P(\text{survive } N \text{ flights}) = P_s \cdot \ldots \cdot P_s = P_s^{\,N} = (1 - P_k)^N$$

Let A be the event that the business woman is killed in a plane crash within the next 20 years. Then

$$\begin{aligned} P(A) &= 1 - P(\text{survive } N \text{ flights}) \\ &= 1 - (1 - P_k)^N \\ &= 1 - \left(1 - \frac{1}{4 \cdot 10^6}\right)^{400} \\ &= 9.9995 \times 10^{-5} \approx \underline{\underline{\frac{1}{10{,}000}}} \end{aligned}$$

Mutually Exclusive Events

Two or more events are said to be mutually exclusive (disjoint), if the occurence of any one of them means the others will not occur. That is, two or more events can not occur at the same time.

Example

Throwing a die, the events '4' and '5' are mutually exclusive.

Facts

- If E_1 and E_2 are mutually exclusive events, then
$$P(E_1 \cap E_2) = 0$$
- If E_1 and E_1 are mutually exclusive events, then
$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Mutually Exclusive Events

Examples

1. Let E_1 be BIS students and E_1 be Arts students in this room. What is $P(E_1 \cap E_2)$ and $P(E_1 \cup E_2)$?
2. Let E_3 be students in the swimming team and E_4 be students in the debating team.
   What is $P(E_3 \cap E_4)$ and $P(E_3 \cup E_4)$?

Solution

1. No overlap, so $P(E_1 \cap E_2) = 0$ and
   $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

2. 

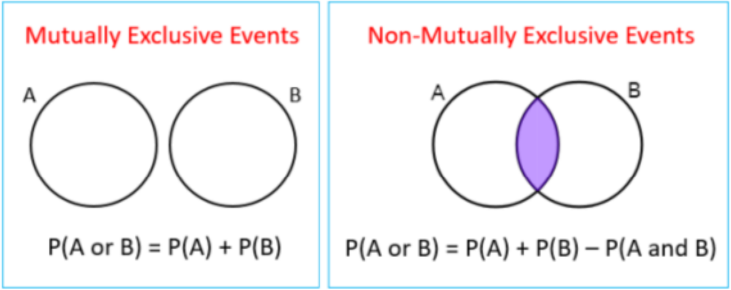
   Pink overlap = students in both teams
   $P(E_3 \cap E_4) = P(\text{Pink overlap})$
   $P(E_3 \cup E_4) = P(E_3) + P(E_4) - P(\text{Pink Overlap})$

Facts

If If E_1 and E_2 are **not mutually exclusive events**, then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

This is best explained with a diagram:



Finding Probabilities

Example

Consider the set of numbers $\{1, 2, 3, ..., 10\}$ and the following events:

- Event A: I pick an even number,
- Event B: I pick a multiple of 3,
- Event C: I pick a multiple of 7.

1. Are events A and B mutually exclusive?
2. What about events A and C?
3. Calculate $P(A \cap B)$ and $P(A \cup B)$.
4. Calculate $P(A \cap C)$ and $P(A \cup C)$.

Finding Probabilities

Solution

$\{1, 2, 3, \dots, 10\}$

$A =$ "even number" $= \{2, 4, 6, 8, 10\}$ $\quad P(A) = \frac{5}{10} = \frac{1}{2}$

$B =$ "multiple of 3" $= \{3, 6, 9\}$ $\quad P(B) = \frac{3}{10}$

$C =$ "multiple of 7" $= \{7\}$ $\quad P(C) = \frac{1}{10}$

1. A, B not disjoint as $6 \in A$ and $6 \in B$.
2. A and C are disjoint.
3. $P(A \cap B) = P(\{6\}) = \frac{1}{10}$

   $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{3}{10} - \frac{1}{10} = \frac{7}{10}$
4. $P(A \cap C) = 0$

   $P(A \cup C) = P(A) + P(C) =$

   $= \frac{1}{2} + \frac{1}{10} = \frac{6}{10} = \frac{3}{5}$

De Morgan's Laws

Rules from set theory can be helpful to calculate probabilities, for example:

De Morgan's Laws

Let A and B be sets. Then

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Example

Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ with subsets $A = \{1, 2, 4, 6, 8, 10\}$ and $B = \{2, 3, 5, 10\}$.
Verify De Morgan's Laws for A and B.

De Morgan's Laws

Solution

- Verify $\overline{A \cup B} = \bar{A} \cap \bar{B}$.

$S = \{1, 2, 3, \dots, 10\}$

$A = \{1, 2, 4, 6, 8, 10\}$ $\quad B = \{2, 3, 5, 10\}$

$\bar{A} = \{3, 5, 7, 9\}$ $\quad \bar{B} = \{1, 4, 6, 7, 8, 9\}$

$A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}$

$\overline{A \cup B} = \{7, 9\}$

$\bar{A} \cap \bar{B} = \{7, 9\} = \overline{A \cup B}$ ✓

- Verify $\overline{A \cap B} = \bar{A} \cup \bar{B}$

This is an exercise for you ü

Finding Probabilities

Example

Suppose we have the following information:

1. There is a 60% chance that it will rain today.
2. There is a 50% chance that it will rain tomorrow.
3. There is a 30% chance that it does not rain either day.

Find the following probabilities:

(a) The probability that it will rain today or tomorrow.
(b) The probability that it will rain today and tomorrow.
(c) The probability that it will rain today but not tomorrow.

Finding Probabilities

Solution

Let $A \hat{=}$ rain today , $B \hat{=}$ rain tomorrow

We have $P(A) = 0.6$, $P(B) = 0.5$, $P(\bar{A} \cap \bar{B}) = 0.3$

(a) $P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{?}$ Need other method

$P(A \cup B) = 1 - P(\overline{A \cup B})$ ← Use De Morgan's Law

$= 1 - P(\bar{A} \cap \bar{B})$

$= 1 - 0.3 = \underline{\underline{0.7}}$

(b) $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.6 + 0.5 - 0.7 = \underline{\underline{0.4}}$

(c) $P(A \cap \bar{B}) = P(A - B)$

$= P(A) - P(A \cap B)$

$= 0.6 - 0.4$

$= \underline{\underline{0.2}}$