MA208 Quantitative Techniques for Business

Lecture 6: Probability

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Today we will first finish our exercises from Monday.

We will then introduce basic concepts from **Probability**:

- Some terminology,
- Probability of an event,
- Conditional probability,
- Independent and dependent events.

Probability is the measure of the likelihood that an event will occur (from Wikipedia).

Revision: exercises from last lecture

1. You are a salesperson for a certain company and next month you must travel to eight destinations in U.S.A.: Dallas, Los Angeles, Miami, Boston, Chicago, Seattle, Omaha, and Kansas City. How many different possible routes can you take?

   8 destinations (order matters)

   \# possible routes $= 8! = 40{,}320$

2. Suppose you plan to invest equal amounts of money in each of five business ventures. If you have 20 ventures from which you make the selection, how many different samples of five ventures can be selected from the 20?

   (order does not matter)

   $$C_5^{20} = \frac{20!}{5!(20-5)!} = 15{,}504$$

Revision: exercises from last lecture

1. In how many ways can the tax office pick five out of 36 companies to audit?

   (order doesn't matter)

   $$C^{36}_{5} = \binom{36}{5} = 376{,}992$$

2. Out of 5 mathematicians and 7 engineers a committee is to be formed that must consist of 2 mathematicians and 3 engineers. In how many ways can this be done?

   $$C^{7}_{2} \cdot C^{5}_{2} = \binom{7}{2}\binom{5}{2} = 350$$

Probability - Why?

Example

Suppose that the NUIG Parking Office claims that two-thirds (67%) of NUIG students maintain a car. Then, suppose we take a random sample of 100 NUIG students and determine that the proportion of students in the sample who maintain a car in NUI Galway is $\frac{69}{100} = 0.69$, that is 69%.

Now we need to ask the question: if the actual population proportion is 0.67, how likely is it that we would get a sample proportion of 0.69?

Any time we want to use a sample to draw a conclusion about some larger population, we need to answer the question "how likely is it...?" or "what is the probability...?". To answer such a question, we need to understand probability, probability rules, and probability models.

Terminology

- An experiment is an observation which has a well-defined set of possible outcomes.
- A sample space is a set of possible outcomes of an experiment.
- An event is a subset of a sample space, i.e. one or several possible outcomes.

Example

1. Suppose we are tossing a coin. The sample space is
$$S = \{\text{Head}, \text{Tail}\} = \{H, T\}.$$
2. Now we are tossing the coin twice. In this case
$$S = \{HH, HT, TH, TT\}.$$

Example

Suppose we randomly select a student, and ask them "how many pairs of jeans do you own?". In this case our sample space S is $S = \{0, 1, 2, 3, ...\}$. Let's define some events.

1. If A is the event that a randomly selected student owns no jeans:
$$E_A = \text{student owns none} = \{0\}.$$
2. If B is the event that a randomly selected student owns some jeans:
$$E_B = \text{student owns some} = \{1, 2, 3, ...\}.$$
3. If C is the event that a randomly selected student owns no more than five pairs of jeans:
$$E_C = \text{student owns no more than five} = \{0, 1, 2, 3, 4, 5\}.$$

Example

Throwing a die.

$S = \{1, 2, 3, 4, 5, 6\}$

Events:

- E_1 = "Even number" $= \{2, 4, 6\}$
- E_2 = "Number 1" $= \{1\}$
- E_3 = "at least 3" $= \{3, 4, 5, 6\}$

Probability of an event

Suppose an event E can happen in r ways out of n possible equally likely ways. Then the probability of occurence of the event (success) is

Probability of an event

$$P(E) = \frac{r}{n}$$

The probability of non-occurence of the event (failure) is

Probability of non-occurence of an event

$$P(\overline{E}) = \frac{n-r}{n} = 1 - \frac{r}{n}$$

Note: in general we have

$$P(E) + P(\overline{E}) = 1$$

Example

See events from Example above.

We have
$$P(E_1) = \frac{3}{6} = \frac{1}{2}$$
$$P(E_2) = \frac{1}{6}$$
$$P(E_3) = \frac{4}{6} = \frac{2}{3}$$

$$P(\text{"not even = odd"}) = P(\bar{E}_1) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(\text{"not 1"}) = P(\bar{E}_2) = 1 - \frac{1}{6} = \frac{5}{6}$$

Examples

- Find the probability of getting tails in a fair coin toss.

$S = \{ head, tail \}$ – 2 possible outcomes
$E = \{ tail \}$ – 1 possible outcome
$P(E) = \frac{1}{2}$

- What is the probability of getting a 5 rolling a die?

$S = \{1, 2, 3, 4, 5, 6\}$
$E = \{5\}$
$P(E) = \frac{1}{6}$

- Tossing a coin three times, what is the probability of getting exactly two heads?

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
$P(\text{"2 heads"}) = \frac{3}{8}$

Examples

- The names of four directors of a company will be placed in a hat and a 2-member delegation will be selected at random to represent the company at an international meeting.
  Let A, B, C and D denote the directors in the company. What is the probability that
  (i) A is selected?
  (ii) A or B is selected?
  (iii) A is not selected?

Possible outcomes $S = \{AB, AC, AD, BC, BD, CD\}$

(i) $P(A) = 3/6 = \frac{1}{2}$

(ii) $P(A \text{ or } B) = 5/6$

(iii) $P(\bar{A}) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$

Conditional Probability

Example

What is the probability of rain, given the forecast was for rain?

Here we are looking for the probability of one event under the condition that another event has already occured. We call this conditional probability.

Conditional Probability

If E_1 and E_2 are two events, the probability that E_2 occurs given that E_1 has occured is denoted by

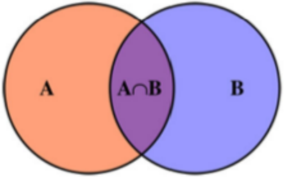
$$P(E_2|E_1)$$

$P(E_2|E_1)$ is called the ***conditional probability of*** E_2 ***given that*** E_1 ***has occured***.

Calculating Conditional Probability

Let E_1 and E_2 be two events defined in a sample space S such that $P(E_1) > 0$. The conditional probability of E_2, assuming that E_1 has already occured, is given by

$$P(E_2|E_1) = \frac{P(E_2 \text{ and } E_1)}{P(E_1)} = \frac{P(E_2 \cap E_1)}{P(E_1)}$$



Calculating Conditional Probability

Let's get back to our example from above.

Example

What is the probability of rain, given the forecast was for rain?

Let A = event "it rains" and B = event "forecast rain".
We are given the following information:

$$P(A \cap B) = 0.7 \quad (1)$$
$$P(B) = 0.8 \quad (2)$$

So

$$P(A|B) = \frac{0.7}{0.8} = \frac{7}{8}$$

Calculating Conditional Probability

Exercise

Let A denote the event "student wears glasses" and let B denote the event "student is French". In a class of 100 students, suppose 60 are French, and suppose that ten of the French students are wearing glasses. Find the probability that if I pick a French student, this student will wear glasses. That is, find $P(A|B)$.

Solution

10 out of 100 students are both French and wear glasses, so

$$P(A \cap B) = \frac{10}{100} = \frac{1}{10}$$

60 out of 100 students are French, so

$$P(B) = \frac{60}{100} = \frac{1}{6}$$

So the required probability is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{10}}{\frac{6}{10}} = \frac{1}{6}$$

Calculating Conditional Probability

Exercise

You toss a fair coin three times:

(i) What is the probability of three heads?

(ii) What is the probability that you observe exactly one heads?

(iii) Given that you have observed at least one heads, what is the probability that you observe at least two heads?

Solution

(i) $2^3 = 8$ possible events : $\{HHH, HHT, \ldots, TTT\}$

$P(HHH) = \frac{1}{8}$

(ii) $P(\text{"exactly one heads"}) = P(HTT \cup THT \cup TTH)$

$= P(HTT) + P(THT) + P(TTH)$

$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

Calculating Conditional Probability

Solution

(iii) Let A_1 = event "at least one heads",

A_2 = event "at least two heads".

Then $A_1 = S - \{TTT\}$ and $P(A_1) = \frac{7}{8}$

$A_2 = \{HHT, HTH, THH, HHH\}$ and $P(A_2) = \frac{4}{8}$

Thus

$$P(A_2 \mid A_1) = \frac{P(A_2 \cap A_1)}{P(A_2)} = \frac{P(A_2)}{P(A_1)} = \frac{4/8}{7/8} = \frac{4}{7}$$