MA208 Quantitative Techniques for Business

Lecture 5: Combinatorics

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Section II: Combinatorics/ Counting Techniques/ Probability

In this Section we will cover

- Combinatorics/ Counting Techniques
- Probability
- Random Variables

Lecture 4 - Outline

To pursue our study more we need to learn some basic principles from Probability, Counting and Combinatorics. An efficient way of counting is necessary to handle large masses of data. Combinatorics is the study of the number of ways a set of objects can be arranged, combined, or chosen. Today we will introduce

- The Additive Principle
- The Multiplicative Principle
- Permutations
- Combinations

Examples

1. The Burger Bar offers the following items on its menu:

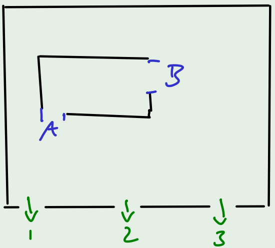
Burger	**Sides**	**Beverages**	**Desserts**
Single Meat	Fries	Tea	Cheesecake
Double Meat	Onion Rings	Coffee	Brownie
	Fruit Bowl	Soda	Cookie
	Mixed Salad		Ice Cream

   (a) If a customer wants only one item, how many choices does s/he have?
   (b) If a customer chooses one item from each category, how many meals can be made?

2. Picture a room in a building. The room has two exits and the building has three exits. In how many ways can we exit the building?

Examples

2



possible ways out:

$$A \begin{cases} \to 1 \\ \to 2 \\ \to 3 \end{cases} + \quad B \begin{cases} \to 1 \\ \to 2 \\ \to 3 \end{cases}$$

$$3 + 3 = \underline{\underline{6}} \text{ (ways)}$$

The Additive Principle

Example

The student union shop offers 4 different NUIG-Hoodies and 7 different NUIG-T-Shirts. You would like to buy a top with NUIG-logo for your friend, an Erasmus student from Hungary who is going home next week.

You can afford only one item, how many choices do you have?

Solution

$4 + 7 = 11$ choices

The Additive Principle

The Additive Principle

If event A can occur in m ways and event B can occur in n (disjoint) ways, then event "A **or** B" can occur in $m + n$ ways.

Example

1. Can we use the additive principle to determine how many two letter "words" begin with either F **or** K?
2. Can we use the additive principle to determine how many two letter "words" contain either F **or** K?

Solution

1. Event "Begin with F" – 26
   Event "Begin with K" – 26
   answer: yes, $26 + 26 = 52$
2. No, not disjoint, for example the word "FK" belongs to both groups

The Multiplicative Principle

The Multiplicative Principle

If event A can occur in m ways and each possibility allows for event B to occur in n (disjoint) ways, then event “A **and** B” can occur in $m \times n$ ways.

Examples

1. When buying a car you can pick four types of engine and six colours. How many choices do you have?
2. How many two letter combinations of the vowels a, e, i, o, u can you find?
   Note: letters are supposed to be different.

Solution

① Event 'Engine' - $m=4$, Event 'Colour' - $n=6$, # choices $= 4\times 6 = 24$

② Letter 1 - 5 options, Letter 2 - 4 options, # combinations $= 5\times 4 = 20$

Factorial notation

$$n! = (n)(n-1)(n-2)\cdot\ldots\cdot(3)(2)(1)$$

Examples

$$\begin{aligned} 5! &= 5 \times 4 \times 3 \times 2 \times 1 \\ 10! &= 10 \times 9 \times 7 \times \ldots \times 3 \times 2 \times 1 \\ 0! &= 1 \quad \text{This is a convention.} \end{aligned}$$

Permutations

An arrangement (or ordering) of a set of objects is called a permutation. In a permutation, the **order** that we arrange the object in, **is important**.

Example

Consider arranging three letters. In how many ways can this be done?

Arranging n objects

In general, n distinct objects can be arranged in $n!$ ways.

Example

In our example above we have

$$3! = 3 \times 2 \times 1 = 6$$

possible ways to arrange the letters.

Permutations

Examples

1. Find all two letter permutations of $\{a, b, c, d\}$.
2. Find all three letter permutations of $\{a, b, c, d\}$.

(1)

ab	ac	ad
ba	bc	bd
ca	cb	cd
da	db	cc

12 combinations

(2)

abc acb bac bca cab cba

abd

acb

bcd

6 comb. of a, b, c
" " of a, b, d
" " of a, c, d
" " of b, c, d

$\Rightarrow 4 \times 6$
$= \underline{\underline{12}}$ combinations

Permutations

Number of Permutations

The number of permutations of n distinct objects taken r at a time, denoted by P_r^n, where repetitions are **not** allowed, is given by

$$P_r^n = (n)(n-1)(n-2)\cdot \dots \cdot (n-r+1) = \frac{n!}{(n-r)!}$$

Note: Other common notations are for example $P(n,r)$ or ${}^nP_r$.

Examples

Use this formula for the examples above.

1. $P_2^4 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = 4 \times 3 = 12$ ✓

2. $P_3^4 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 4! = 4 \times 3 \times 2 = 24$ ✓

Combinations

A combination of n objects taken r at a time is a selection which does not take into account the arrangements of the objects. That is, the **order** is **not important**.

Example

Students are asked to choose four different letters from the English alphabet. David selects $\{A, E, R, T\}$, Sarah selects $\{D, E, N, Q\}$ and Anna selects $\{R, E, A, T\}$.

1. How many sets of four letters have the three students chosen together?

   2 different sets

2. How many different sets of four letters can be selected from the alphabet?

   Write down sets ... or use the following:

Combinations

Number of Combinations

The number of ways (or combinations) in which r objects can be selected from a set of n objects, where repetition is **not** allowed, is

$$C_r^n = \frac{n!}{r!(n-r)!}$$

Note: C_r^n is often denoted by $\binom{n}{r}$ ("n choose or r") or ${}^nC_r$.

Example

Use this formula to calculate the number of different sets of four letters which can be chosen from the alphabet.

$$C_4^{26} = \binom{26}{4} = \frac{26!}{4!(26-4)!} = \frac{26!}{4!\,22!} = \underline{14,950}$$

Exercises

1. You are a salesperson for a certain company and next month you must travel to eight destinations in U.S.A.: Dallas, Los Angeles, Miami, Boston, Chicago, Seattle, Omaha, and Kansas City. How many different possible routes can you take?
2. Suppose you plan to invest equal amounts of money in each of five business ventures. If you have 20 ventures from which you make the selection, how many different samples of five ventures can be selected from the 20?
3. In how many ways can the tax office pick five out of 36 companies to audit?
4. Out of 5 mathematicians and 7 engineers a committee is to be formed that must consist of 2 mathematicians and 3 engineers. In how many ways can this be done?

Exercises

Q1. eight destinations (order matters)

$\#$ of possible routes $= 8! = 40{,}320$

Have a go at the other questions yourself!

We'll get back to them on Friday.