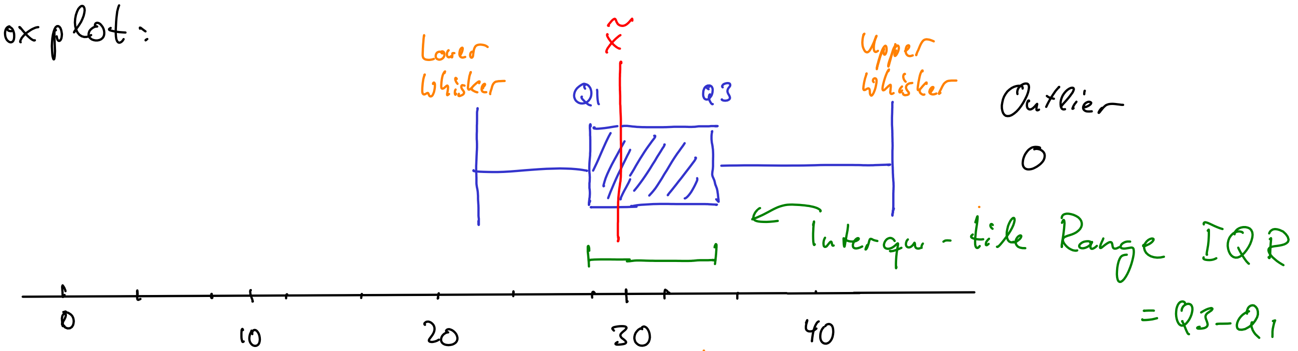
Lecture 24 Revision

Statistics.

Revise
- histogram, stem & leaf plot
- median $\tilde{x}$
- 5 number summary = { min, Q_1, median, Q_3, max }

  where Q_1 = median of data below the median,
  Q_3 = " " " above " "

- Boxplot:



The median cuts the box into two pieces.

- If fewer data plot to the right, the data is skewed right,
- " " " " " " left, " " " skewed left.

So in our example the boxplot is left-skewed.

The shape of the boxplot helps to choose the best measure of centrality or variance.

- The median / IQR are the best measure of centrality / variance for skewed distributions or data with strong outliers.

- The mean / standard deviation are the best measure of centrality / variance for symmetric distributions with no outliers.

Exercise

Find an example of a data set where the median is a better measure of the central tendency of the data than the mean.

- (arithmetic) mean $\bar{x} = \dfrac{\sum x_i}{n}$

Ex: Calculate the mean of $\{30, 22, 44, 31, 28, 28\}$

$$\bar{x} = \frac{30 + 22 + 44 + 31 + 28 + 28}{6}$$
$$= 30.5$$

- harmonic mean $= \left(\dfrac{\sum \frac{1}{x_i}}{n} \right)^{-1}$

(used for average of rates)

- geometric mean $= \left(\prod x_i \right)^{\frac{1}{n}}$

(used when comparing different items)

- weighted mean

- standard deviation (another measure of spread, often better than midrange or IQR)

$$\sigma = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}$$

- coefficient of variation (cv) $= \frac{\sigma}{\bar{x}} \cdot 100\%$

Chebyshev's inequality:

At least $1 - \frac{1}{k^2}$ of the distribution's values are within k standard deviations of the mean, $k \geq 1$.

Example:

The amount of soft drink (in ounces) to be filled in bottles has a mean of μ ounces and a standard deviation of σ ounces.
The quality control engineer at the bottling plant desires the amount of soft drink X to be within 1 ounc of the mean at least 90% of the time.
If the quality control engineer's goals are to be met, what is the largest value of σ that can be tolerated?

Solution:

$$90\% \text{ in } (\mu - k\sigma, \mu + k\sigma)$$
$$= (\mu - 1, \mu + 1)$$
$$\text{so } k\sigma = 1$$

$$1 - \frac{1}{k^2} = \frac{9}{10}$$
$$\frac{k^2 - 1}{k^2} = \frac{9}{10}$$
$$k^2 - 1 = \frac{9}{10} k^2$$
$$\frac{1}{10} k^2 = 1 \quad \Rightarrow \quad k^2 = 10 \quad \Rightarrow \quad k = \sqrt{10}$$
$$k\sigma = 1 \quad \Rightarrow \quad \sigma = \frac{1}{\sqrt{10}} = \underline{0.31623}$$

Probability

Exam 2013/14, Q C.

(a) X = number of heads

2 possible outcomes T/F

binomial distribution, so

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x},$$

with $n = 5$, $p = \frac{1}{2}$

$$P(X=0) = \binom{5}{0} \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^5 = \left(\tfrac{1}{2}\right)^5 = \frac{1}{32}$$

$$P(X=1) = \binom{5}{1} \left(\tfrac{1}{2}\right)^1 \left(\tfrac{1}{2}\right)^4 = 5 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2^4} = \frac{5}{32}$$

$$P(X=2) = \binom{5}{2} \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^3 = 10 \cdot \tfrac{1}{2^2} \cdot \tfrac{1}{2^3} = \frac{10}{32}$$

$$P(X=3) = \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = 10 \cdot \frac{1}{2^3} \cdot \frac{1}{2^2} = \frac{10}{32}$$

$$P(X=4) = \binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1 = 5 \cdot \frac{1}{2^4} \cdot \frac{1}{2} = \frac{5}{32}$$

$$P(X=5) = \binom{5}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0 = 1 \cdot \frac{1}{2^5} = \frac{1}{32}$$

(b) If even number of coins show heads $(X = 2, 4)$

$\Rightarrow$ win €X^2+X

If odd number of coins show heads $(X = 1, 3, 5)$

$\Rightarrow$ lose €X^2+X

$X=0 \Rightarrow$ no win $(X^2+X=0)$

So expected winnings are

$$P(X=0)\cdot 0 + P(X=1)\left(-(1^2+1)\right) + P(X=2)\left(+(2^2+2)\right)$$

$$+ P(X=3)\left(-(3^2+3)\right) + P(X=4)\left(+(4^2+4)\right) + P(X=5)\left(-(5^2+5)\right)$$

$$= \frac{1}{32}(0) + \frac{5}{32}(-2) + \frac{10}{32}(6) + \frac{10}{32}(-12) + \frac{5}{32}(20) + \frac{1}{32}(-30)$$

$$= -\frac{10}{32} + \frac{60}{32} - \frac{120}{32} + \frac{100}{32} - \frac{30}{32}$$

$$= 0$$

So expected winnings $= 0$

$\Rightarrow$ No point to play the game if you are interested in winning money!

Note: You could have written the calculation as

$$\sum_{n=0}^{5} P(X=n)\,(-1)^n\,(n+n^2)$$

(c) Here we have three events for each of the companies: Value when ruble exchange rate is sinking (x_1), rising (x_2) and stagnating (x_3), with probabilities as follows:

$$P(x_1 = \text{sinking}) = 20\,\% = 0.2$$
$$P(x_2 = \text{rising}) = 20\,\% = 0.2$$
$$P(x_3 = \text{stagnating}) = 60\,\% = 0.6$$

(i) **Recall:** Expected value $E(x) = \sum x_i\, P(x_i)$

- Nickel: $E(x) = (0.2)(150) + (0.2)(20) + (0.6)(-50)$
  $= €4$
- Gazprom: $E(x) = (.2)(-20) + (.2)(130) + (.6)(-10)$
  $= €16$
- Aeroflot: $E(x) = (.2)(20) + (.2)(30) + (.6)(52)$
  $= €52$

So Aeroflot has the best expected shareholder value.

(ii) The expected value of the informed choice is

probability of striking   prob. of rising   prob. of stagn.

$$E(x) = (.2)(150) + (.2)(130) + (.6)(70)$$

best choice   best choice   best choice

$$= €\,98,$$

which is $(98 - 52) = €\,46$ better

$\Rightarrow$ Good value for information!

Finance Maths

Exam 2015, Q 3.

(Formulae provided in exam)

(a) Use Simple Interest Formula

$$A = P(1 + rt)$$

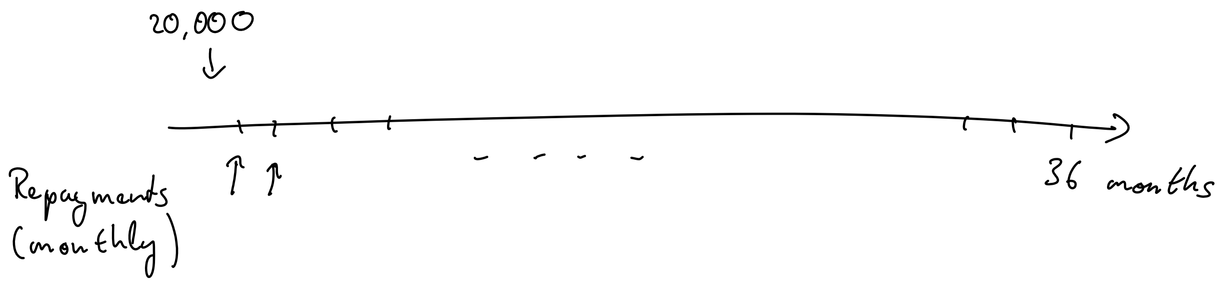
here $A = €\,10{,}000$, $r = 0.04$, $t = 3$,
looking for P.

$$10{,}000 = P\,(1 + (0.04)(3))$$

$$10{,}000 = P\,(1.12)$$

$$\Rightarrow \quad P = \frac{10{,}000}{1.12}$$

(b) Annuity with compounded interest 12% p.a. comp mont



Use Present Value Formula :

$$PV = PMT \, \frac{1-(1+i)^{-n}}{i}$$

with $PV = 20{,}000$, $i = \frac{0.12}{12} = 0.01$, $n = 36$

looking for PMT

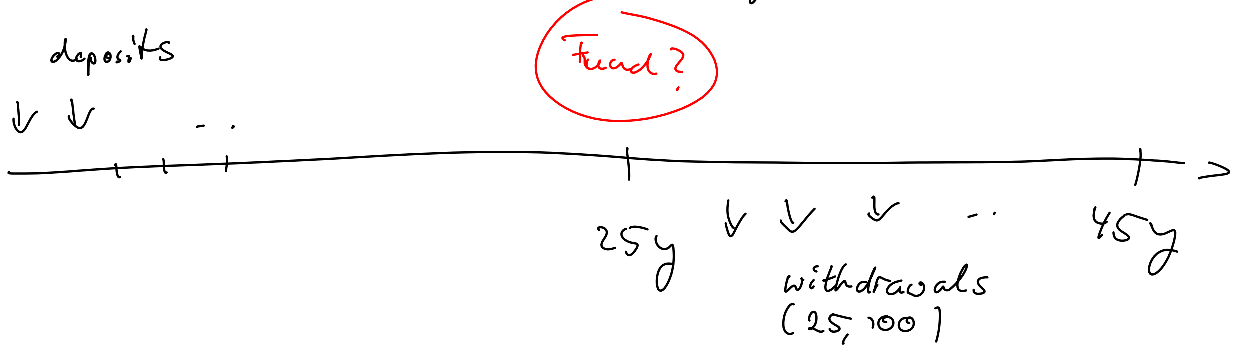
$$20{,}000 = PMT \, \frac{1-(1+0.01)^{-36}}{0.01}$$

$$\Rightarrow \quad PMT = 20{,}000 \, \frac{0.01}{1-(1.01)^{-36}}$$

$$= € \, 664.29$$

(c)

6.5% compounded annually



(i) Use

PV – formula: $PV = PMT \dfrac{1-(1+i)^{-n}}{i}$

to calculate the fund.

$PMT = 25{,}000$, $i = 0.065$, $n = 20$

$$(\text{Fund} =)\ PV = 25{,}000 \dfrac{1-(1+0.065)^{-20}}{0.065}$$

$$= € \underline{\underline{275{,}462.68}}$$

(ii) Use Future value formula

$FV = PMT \dfrac{(1+i)^n - 1}{i}$

to calculate amount of each deposit.

Here $FV = € 275{,}462.68$, $i = 0.065$, $n = 25$

So $275{,}462.68 = PMT \dfrac{(1.065)^{25}-1}{0.065}$

$$\Rightarrow (\text{each deposit} =)\ PMT = 275{,}462.68 \dfrac{0.065}{(1.065)^{25}-1}$$

$$= \underline{€ 4{,}677.76}$$

(iii) Total deposits $= 25 \times 4{,}677.76 = €\,116{,}944$

Total withdrawals $= 20 \times 25{,}000 = €\,500{,}000$

Total interest earned $= €\,500{,}000 - €\,116{,}944$

$= \underline{\underline{€\,383{,}056}}$

Linear Programming

Eg. An oil company wish to inject a fracking fluid into the ground of an island and can use two types of chemicals, Bromine and DBNPA. Both types have an impact on the environment and therefore there are limits on the impact type:

Impact type	units extracted with either chemical: Bromine	DBNPA	Limits
Drinking water	40 kTpu	20 kTpu	1,200 kilotons
Air pollution	30 "	30 "	1,800 "
Surface contamination	25 "	50 "	1,500 "

(kpTu = kilotons per unit)

Let x and y denote the number of units extracted with Bromine (resp. DBNPA).

Find the region of feasible solutions and corner points.

$$\begin{aligned}
&① \quad 40x + 20y \le 1{,}200\\
&② \quad 30x + 30y \le 1{,}800\\
&③ \quad 25x + 50y \le 1{,}500
\end{aligned}$$

- Find lines.

① $40x + 20y = 1{,}200$

$\Rightarrow 2x + y = 60$

$x = 0 \Rightarrow y = 60 \longrightarrow (0, 60)$

$y = 0 \Rightarrow x = 30 \longrightarrow (30, 0)$

② $30x + 30y = 1{,}800$

$\Rightarrow x + y = 60$

$x = 0 \Rightarrow y = 60 \qquad (0, 60)$

$y = 0 \Rightarrow x = 60 \qquad (60, 0)$

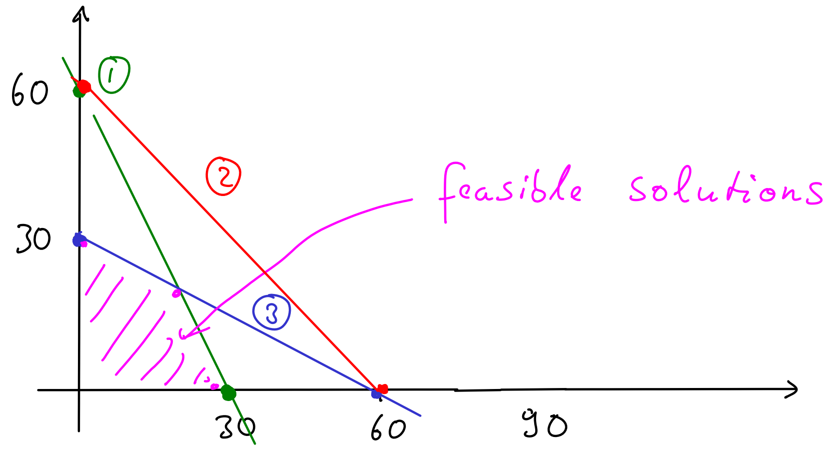
③ $25x + 50y = 1{,}500$

$\Rightarrow x + 2y = 60$

$x = 0 \Rightarrow y = 30 \qquad (0, 30)$

$y = 0 \Rightarrow x = 60 \qquad (60, 0)$

- Sketch the graph and shade the feasible region.



- Find the corner points.

Corner points are $(0,30)$, $(30,0)$ and the intersection of line 1 and 3.

Intersection of 1 and 3:

1 $\quad 2x + y = 60 \Rightarrow y = -2x + 60$

3 $\quad x + 2y = 60 \Rightarrow y = -\frac{1}{2}x + 30$

$(y=y)$ $\quad -2x + 60 = -\frac{1}{2}x + 30$

$$\Rightarrow -\frac{3}{2}x = -30$$

$$\Rightarrow x = 20$$

If $x = 20$, then $y = -2(20) + 60 = 20$

$\therefore$ The third corner point is $(20, 20)$.

Eg (ctd) The company makes a profit of € 13 million using Bromine and € 7 million using DPNPA. Determine the company's maximum profit.

Profit function $p = 13x + 7y$.

- Substitute values from corner points.

- $(30, 0)$ $p = 13(30) + 7(0) = 390$
- $(0, 30)$ $p = 13(0) + 7(30) = 210$
- $(20, 20)$ $p = 13(20) + 7(20) = 400$ ... best!

$\therefore$ The company makes a maximum profit of € 400 million if they use 20 kilotons of Bromine and 20 kilotons of DBNPA per unit.