MA208 Quantitative Techniques for Business

Lecture 23: Linear Programming ctd. - The Simplex Method

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Lecture 23

Last week we have formulated a linear programming model for the *Craic & Co.* Example, but didn't have enough time to solve it. We will do that today - by using the Simple Method.

Let's remind ourselves of the example.

The *Craic & Co.* Company Example

Craic & Co. produce both interior and exterior paints from two raw materials, $M1$ and $M2$. The following table provides the basic data of the problem:

	per ton of Exterior Paint	per ton of Interior Paint	Max. daily availability
Raw Material, $M1$	6 tons	4 tons	24 tons
Raw Material, $M2$	1 tons	2 tons	6 tons
Profit per ton (€1000)	5	4	

They also know that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. Also, the maximum daily demand of interior paint is 2 tons.

Craic & Co. want to determine the optimum (best) product mix of interior and exterior paints that maximises the total daily profit.

Formulating the Model

Let x be the tons produced daily of exterior paints and y be the tons produced daily of interior paints. Then we have the following constraints:

$$\begin{aligned} 6x + 4y &\leq 24 \\ x + 2y &\leq 6 \\ y - x &\leq 1 \\ y &\leq 2 \end{aligned}$$

as well as nonnegativity restrictions $x \geq 0$, $y \geq 0$.

The objective of the company is to maximise

$$p = 5x + 4y.$$

Setting up for the Simplex Method

The right-hand side of the inequality constraints can be thought of as representing the limit on the availability of a resource, in which case the left-hand side would represent the usage of this limited resource by the activities (variables) of the model. The difference between the right-hand side and the left-hand side of the inequality constraints thus yields the unused or **slack** amount of the resource.

The first constraint in our example

$$6x + 4y \leq 24$$

can be converted to

$$6x + 4y + s_1 = 24, \quad s_1 \geq 0.$$

The Simplex Algorithm

Similarly, we can convert the *Craic & Co.* problem as follows.

$$
\begin{aligned}
6x &+4y + s_1 & & & &= 24\\
x &+2y & +s_2 & & &= 6\\
-x &+y & & +s_3 & &= 1\\
 & y & & & + s_4 &= 2
\end{aligned}
$$

The variables s_1, s_2, s_3, s_4 are the slacks associated with the respective constraints.

Next we express the objective function as

$$p - 5x - 4y = 0$$

The Simplex Algorithm

In this manner the starting tableau can be represented as follows:

Basic	p	x	y	s_1	s_2	s_3	s_4	Solution
p	1	-5	-4	0	0	0	0	0
s_1	0	6	4	1	0	0	0	24
s_2	0	1	2	0	1	0	0	6
s_3	0	-1	1	0	0	1	0	1
s_4	0	0	1	0	0	0	1	2

We now need to manipulate the equations in the last tableau so that the Basic-column and the Solution-column will identify the new solution.
This is achieved through Gauss-Jordan row operations (board work).

The Simplex Algorithm

Starting tableau:

	p	x	y	s_1	s_2	s_3	s_4	Solution
p	1	-5	-4	0	0	0	0	0
s_1	0	6	4	1	0	0	0	24
s_2	0	1	2	0	1	0	0	6
s_3	0	-1	1	0	0	1	0	1
s_4	0	0	1	0	0	0	1	2

$R2 \leftarrow \frac{1}{6} R2$

	p	x	y	s_1	s_2	s_3	s_4	Solution
p	1	-5	-4	0	0	0	0	0
s_1	0	$\boxed{1}$	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	4
s_2	0	1	2	0	1	0	0	6
s_3	0	-1	1	0	0	1	0	1
s_4	0	0	1	0	0	0	1	2

$R1 \leftarrow R1 + 5R2$

$R3 \leftarrow R3 - R2$

$R4 \leftarrow R4 + R2$

$R3 \leftarrow \frac{3}{4} R3$

	p	x	y	s_1	s_2	s_3	s_4	Solution
p	1	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	20
x	0	1	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	4
s_2	0	0	$\frac{4}{3}$	$-\frac{1}{6}$	1	0	0	2
s_3	0	0	$\frac{5}{3}$	$\frac{1}{6}$	0	1	0	5
s_4	0	0	1	0	0	0	1	2

The Simplex Algorithm

The new tableau corresponding to the new basic solution (x, s_2, s_3, s_4) thus becomes

Basic	p	x	y	s_1	s_2	s_3	s_4	Solution
p	1	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	20
x	0	1	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	4
s_2	0	0	$\frac{4}{3}$	$-\frac{1}{6}$	1	0	0	2
s_3	0	0	$\frac{5}{3}$	$\frac{1}{6}$	0	0	1	5
s_4	0	0	1	0	0	0	1	2

The Simplex Algorithm

Continuing Gauss-Jordan row operations gives the tableau corresponding to the new basic solution (x, y, s_3, s_4).

	p	x	y	s_1	s_2	s_3	s_4	Solution	
p	1	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	20	$R1 \leftarrow R1 + \frac{2}{3} R3$
x	0	1	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	4	$R2 \leftarrow R2 - \frac{2}{3} R3$
s_2	0	0	①	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$	
s_3	0	0	$\frac{5}{3}$	$\frac{1}{6}$	0	1	0	5	$R4 \leftarrow R4 - \frac{5}{3} R3$
s_4	0	0	1	0	0	0	1	2	$R5 \leftarrow R5 - R3$

	p	x	y	s_1	s_2	s_3	s_4	Solution
p	1	0	0	$\frac{3}{4}$	$-\frac{1}{2}$	0	0	3
x	0	1	0	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
y	0	0	①	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
s_3	0	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
s_4	0	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

The Simplex Algorithm

We get the tableau corresponding to the new basic solution (x, y, s_3, s_4):

Basic	p	x	y	s_1	s_2	s_3	s_4	Solution
p	1	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
x	0	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
y	0	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
s_3	0	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
s_4	0	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

Because none of the p-row coefficients associated with the nonbasic variables s_1 and s_2 are negative, this last tableau is optimal

The Simplex Algorithm

The optimum solution can be read from the simplex tableau in the following manner.

The optimum values of the variables in the *Basic*-column are given in the right-hand-side *Solution*-column and can be interpreted as

Decision variable	Optimum value	Recommendation
x	3	Produce 3 tons of exterior paint daily
y	1.5	Produce 1.5 tons of interior paint daily
p	21	Daily profit €21,000