

# MA208 Quantitative Techniques for Business

## Lecture 19: Using Gaussian Elimination to solve a linear system

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## Recap

In Lecture 16 we introduced *systems of linear equations* and revised methods to solve them efficiently by eliminating variables.

In Lectures 17/18 we showed how systems of linear equations can be represented by *matrices*. Many problems can be *expressed* in terms of matrices, and *solved* by *operations* on matrices. Matrices can be *added* and *multiplied* (provided they have matching sizes). *Elementary row operations* can be used to find the inverse of a **square** matrix, and the inverse matrix can be used to solve *systems of linear equations*, providing they have the same amount of variables and equations.

## Lecture 19

Today we will learn how *elementary row operations* can be used to solve **any** systems of linear equations: we'll introduce an algorithm called ***Gaussian Elimination***.

Let's look at an example to introduce this algorithm.

Consider the following linear system:

(i)  $a + b + c = 4$

(ii)  $8a + 2b + c = -1$

(iii)  $-a + b + c = 2$

This set of equations is represented by the **augmented matrix**

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 8 & 2 & 1 & -1 \\ -1 & 1 & 1 & 2 \end{pmatrix}$$

By a sequence of elementary row operations this matrix can be changed to

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{7}{6} & \frac{33}{6} \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

We will show later how this transformation has been achieved.

Comparing this with the original set of equations it is clear that the third row of the matrix represents the equation

$$c = 3.$$

Substitution of this value of  $c$  into the equation represented by Row 2 gives

$$b + \frac{7}{6}c = \frac{33}{6}$$
$$b = \frac{33}{6} - \frac{21}{6} = 2.$$

Finally Row 1 of the simplified matrix represents the equation

$$a + b + c = 4$$

$$a + 2 + 3 = 4$$

$$a + 5 = 4$$

$$a = -1$$

Thus the solution to the linear system is  $a = -1, b = 2, c = 3$ .

The simplified matrix

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{7}{6} & \frac{33}{6} \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

is said to be in the ***row echelon form***.

It represents the same linear system as the original matrix, but it enables us to arrive at the solution to the linear system in just a few calculations as seen before.

We will now learn how to reduce the matrix to row-echelon form by using *Gaussian Elimination*.

Solve the linear system

$$\begin{aligned} a + b + c &= 4, \\ 8a + 2b + c &= -1, \\ -a - b + c &= 2. \end{aligned}$$

First, represent the linear system by the augmented matrix:

$$\left( \begin{array}{cccc} 1 & 1 & 1 & 4 \\ 8 & 2 & 1 & -1 \\ -1 & -1 & 1 & 2 \end{array} \right)$$

Now reduce the matrix to *row echelon form* by using *Gaussian Elimination*.

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 8 & 2 & 1 & -1 \\ -1 & -1 & 1 & 2 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 8R_1, R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & -6 & -7 & -33 \\ 0 & 0 & 2 & 6 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow -\frac{1}{6}R_2, R_3 \leftarrow +\frac{1}{2}R_3} \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{7}{6} & \frac{33}{6} \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

From here we get the solution of the linear system:

- by **back-substitution** (as we did in the beginning of the lecture) or
- by finding the **reduced row echelon form** (eliminating values above the main diagonal).

**Note:** In this course we will use back-substitution.

### Example

A corporation wants to lease a fleet of 12 airplanes with a combined carrying capacity of 220 passengers. The three available types of planes carry 10, 15 and 20 passengers, respectively. How many of each type of plane could be leased?

# Example

## Solution

Let the three types of planes be  $x, y, z$  respectively.

Then we have  $x + y + z = 12$  and  
 $10x + 15y + 20z = 220$ .

$$\left( \begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 10 & 15 & 20 & 220 \end{array} \right) \quad R_2 \leftarrow R_2 - 10R_1$$

$$\left( \begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 5 & 10 & 100 \end{array} \right) \quad R_2 \leftarrow \frac{1}{5}R_2 \quad [\text{Row Echelon Form}]$$

$$\left( \begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 2 & 20 \end{array} \right) \quad R_1 \leftarrow R_1 - R_2$$

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & -8 \\ 0 & 1 & 2 & 20 \end{array} \right) \quad [\text{Reduced Row Echelon Form}]$$

## Solution

$$R1 \Rightarrow x - z = -8$$

$$R2 \Rightarrow y + 2z = 20$$

$x, y, z$  need to be non-negative integers.

$$\left. \begin{array}{l} x = z - 8 \geq 0 \\ y = -2z + 20 \geq 0 \end{array} \right\} \Rightarrow \begin{array}{l} z \geq 8 \text{ and} \\ z \leq 10 \end{array}$$

$$\text{So } 8 \leq z \leq 10$$

We get three possible solutions:

$x$	0	1	2
$y$	4	2	0
$z$	8	9	10

# Example

## Example

The University of Tennessee Lady Volunteers defeated the Rutgers University Scarlet Knights 59 to 46. The Lady Volunteers' scoring resulted from a combination of three-point baskets, two-point baskets, and one-point free throws. There were three times as many two-point baskets as three-point baskets. The number of free throws was one less than the number of two-point baskets. (Source: American National Collegiate Athletic Association)

- (a) Set up a system of linear equations to find the numbers of three-point baskets, two-point baskets, and one-point free throws scored by the Lady Volunteers.
- (b) Solve your system.

## Solution

Let  $x$  be the number of three-point baskets,  
 $y$  " " " " two-point " ,  
 $z$  " " " " one-point free throws.

Then we have

$$\begin{aligned} 3x + 2y + z &= 59 \\ 3x - y &= 0 \\ y - z &= 1 \end{aligned}$$

The augmented matrix is

$$\left( \begin{array}{ccc|c} 3 & 2 & 1 & 59 \\ 3 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{array} \right)$$

## Solution

Reduce to Row Echelon Form

$$\left( \begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & \frac{59}{3} \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 14 \end{array} \right)$$

Using Back - Substitution we get

$$x = 5, \quad y = 15, \quad z = 14$$

The Lady Volunteers scored 5 three-point baskets, 15 two-point baskets and 14 one-point free throws.