MA208 Quantitative Techniques for Business

Lecture 18: Systems of linear equations, Matrices ctd.

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Lecture 18 - Outline

Today we will

- Use the inverse matrix to solve a resource allocation problem.

Finding the inverse of an n x n matrix

Last week we learned a formula to calculate the inverse of a 2×2 matrix and the *Gauss Jordan method* to calculate the inverse of an $n \times n$ matrix.
This method works for any $n \times n$ matrix, but it takes time to calculate the inverse of a large matrix.

However, sometimes we are lucky. Consider the following example ...

Example

Let $A = \begin{pmatrix} 3 & 4 & 5 \\ 3 & 2 & 1 \\ 2 & 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 8 & -6 \\ -7 & -1 & 12 \\ 8 & -4 & -6 \end{pmatrix}$.

Compute $A \cdot B$.

Example

Solution

$$A \cdot B = \begin{pmatrix} 3 & 4 & 5 \\ 3 & 2 & 1 \\ 2 & 4 & 3 \end{pmatrix} \begin{pmatrix} 2 & 8 & -6 \\ -7 & -1 & 12 \\ 8 & -4 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{pmatrix}$$

$$= 18 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 18 \cdot I$$

Example

So we have

$$A \cdot B = 18\, I$$

or

$$A \cdot \frac{1}{18} B = I$$

Recall: If $A \cdot M = I$, then $M = A^{-1}$.

Thus we get

$$\frac{1}{18} B \;=\; A^{-1} \qquad (*)$$

We can use this to solve the following problem. How?

Example

Example

A bakery wants to produce three types of bread. Each morning, one bag of wheat flour, one bag of rye flour and one bag of barley flour are opened. The required quantities of each flour for one loaf of bread in each of the three recipes are specified in the table below, in terms of multiples of the measuring cup that is used for taking flour out of the bags.

Grain type	Bread A	Bread B	Bread C	Size of each flour bag
Wheat	3 cups	4 cups	5 cups	86 cups
Rye	3 cups	2 cups	1 cup	40 cups
Barley	2 cups	4 cups	3 cups	64 cups

Example

Example (ctd.)

Grain type	Bread A	Bread B	Bread C	Size of each flour bag
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(i) Let x, y and z be the numbers of bread loaves A, B and C produced in one morning. Write down a system of three linear equations which hold precisely when all the three resources are fully used.

(ii) Find the values of x, y and z which ensure that the bags of flour are fully used.

Example

Solution

(i) $x \hat{=}$ number of bread loaves 'A'
$y \hat{=}$ " " " " 'B'
$z \hat{=}$ " " " " 'C'

Then
$$3x + 4y + 5z = 86$$
$$3x + 2y + z = 40$$
$$2x + 4y + 3z = 64$$

(ii) As matrix equation:
$$\begin{pmatrix} 3 & 4 & 5 \\ 3 & 2 & 1 \\ 2 & 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 86 \\ 40 \\ 64 \end{pmatrix}$$
$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 86 \\ 40 \\ 64 \end{pmatrix} \quad \text{with } A = \begin{pmatrix} 3 & 4 & 5 \\ 3 & 2 & 1 \\ 2 & 4 & 3 \end{pmatrix}$$

We know from the example above (see (*)) that
$$A^{-1} = \frac{1}{18} \begin{pmatrix} 2 & 8 & -6 \\ -7 & -1 & 12 \\ 8 & -4 & -6 \end{pmatrix}$$

Example

Solution

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 86 \\ 40 \\ 64 \end{pmatrix}$$

$$A^{-1} A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 86 \\ 40 \\ 64 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 86 \\ 40 \\ 64 \end{pmatrix}$$

$$= \frac{1}{18} \begin{pmatrix} 2 & 8 & -6 \\ -7 & -1 & 12 \\ 8 & -4 & -6 \end{pmatrix} \begin{pmatrix} 86 \\ 40 \\ 64 \end{pmatrix}$$

$$= \frac{1}{18} \begin{pmatrix} 172 + 320 - 384 \\ -602 - 40 + 768 \\ 688 - 160 - 384 \end{pmatrix}$$

$$= \frac{1}{18} \begin{pmatrix} 108 \\ 126 \\ 144 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 8 \end{pmatrix}$$

So $x = 6$, $y = 7$, $z = 8$.

Solving a system of *any* size

So far we have learned how *elementary row operations* can be used to find the inverse of a **square** matrix, and the inverse matrix can be used to solve *systems of linear equations*, providing they have the same amount of variables and equations.
Next week we will learn how *elementary row operations* can be used to solve **any** systems of linear equations: we'll introduce an algorithm called *Gaussian Elimination*.

Now, work on Questions 8 and 9 from Problem Sheet 4. :)