MA208 Quantitative Techniques for Business

Lecture 13: Mathematics of Finance ctd.

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Lecture 13 - Outline

Today we will talk about

- Sinking Funds
- Present value of an Annuity

Revision: Future Value of an Annuity

Example

What is the value of an annuity at the end of 20 years if €2,000 is deposited each year into an account earning 8.5% p/a compounded annually?

Solution

$FV = ?,\quad PMT = 2{,}000,\quad i = 0.085,\quad n = 20$

$$FV = 2000\,\frac{(1+0.085)^{20}-1}{0.085}$$

$$= €96{,}754.03$$

Sinking Funds

Any account that is established to accumulate funds to meet future obligations or debts is called a sinking fund. If the payments are to be made in the form of an ordinary annuity we only need to solve the annuity formula for the payment PMT.

Example

The parents of a newborn child wish to deposit € x every year into an account that pays 6% compounded annually. Compute x, if they want € $80,000$ for when the child is 17.

Solution

$FV = 80{,}000$, $PMT = x$, $i = 0.06$, $n = 17$

$$80{,}000 = x \frac{(1+0.06)^{17} - 1}{0.06}$$

$$x = \frac{80{,}000 \cdot (0.06)}{(1.06)^{17} - 1} = € \ 2{,}835.58$$

Sinking Funds

Example

A company estimates that it will need to replace a piece of equipment at a cost of €50,000 in 5 years. To have this money available in 5 years it establishes a sinking fund by making equal monthly payments into an account paying 6.6% p/a compounded monthly.

(i) How much should each payment be?

(ii) How much interest is earned during the last year?

Solution

(i) $FV = 50{,}000$, $i = \frac{0.066}{12} = 0.0055$, $n = 60$, $PMT = ?$

$$50{,}000 = PMT \, \frac{(1+0.0055)^{60} - 1}{0.0055}$$

$$PMT = €\,705.65 \text{ per month}$$

Sinking Funds

Solution

(ii) • Amount after 4 years:

PMT = 705.65, $i = 0.0055$, $n = (4)(12) = 48$

$$FV = 705.65\,\frac{(1.0055)^{48}-1}{0.0055} = €\,38{,}642.25$$

• Growth in 5th year:

50,000 − 38,642.25 = 11,357.75

• Payments during 5th year:

12 × 705.65 = 8,467.80

• Interest earned during 5th year:

11,357.75 − 8,467.80 = €2,889.95

Present value of an Annuity

Example

How much should you deposit in an account paying 6% compounded semiannually in order to be able to withdraw €1000 every six months for the next three years?

Note: After the last payment is made, no money is to be left in the account.

Present value of an Annuity

Here we are interested in finding the present value of each €1000 that is paid out during the three years.

We can find this by using the compound interest formula and solving for P:

$$
\begin{aligned}
A &= P(1+i)^n \\
P &= \frac{A}{(1+i)^n} = A(1+i)^{-n},
\end{aligned}
$$

where the rate per period is $i = \frac{0.06}{2} = 0.03$.

Present value of an Annuity

The present value for the **first** payment is

$$P_1 = 1000\,(1.03)^{-1}$$

The present value for the **second** payment is

$$P_2 = 1000\,(1.03)^{-2}$$

...

The present value for the **sixth** payment is

$$P_6 = 1000\,(1.03)^{-6}$$

So the present value **after** 3 **years** (i.e. 6 payments) is

$$P = 1000(1.03)^{-1} + 1000(1.03)^{-2} + ... + 1000(1.03)^{-6} \quad \textbf{(1)}$$

Present value of an Annuity

Multiplying equation **(1)** by 1.03, we get

$$1.03P = 1000 + 1000(1.03)^{-1} + ... + 1000(1.03)^{-5} \qquad \textbf{(2)}$$

Now we subtract equation **(1)** from equation **(2)**:

$$\begin{aligned} 1.03P - P &= 1000 - 1000(1.03)^{-6} \\ 0.03P &= 1000 - 1000(1.03)^{-6} \\ P &= \frac{1000 - 1000(1 + 0.03)^{-6}}{0.03} \end{aligned}$$

Present value of an Annuity

Present value of an Annuity

In general the Present Value of an Ordinary Annuity is

$$PV = PMT\,\frac{(1-(1+i)^{-n})}{i}$$

where PV = present value of all payments, PMT = periodic payment, i = rate per period, and n = number of periods.

Note: Payments are made at the end of each period.

So in our example, we get

$$PV = 1000\,\frac{(1-(1.03)^{-6})}{0.03} = €\,5417.19.$$