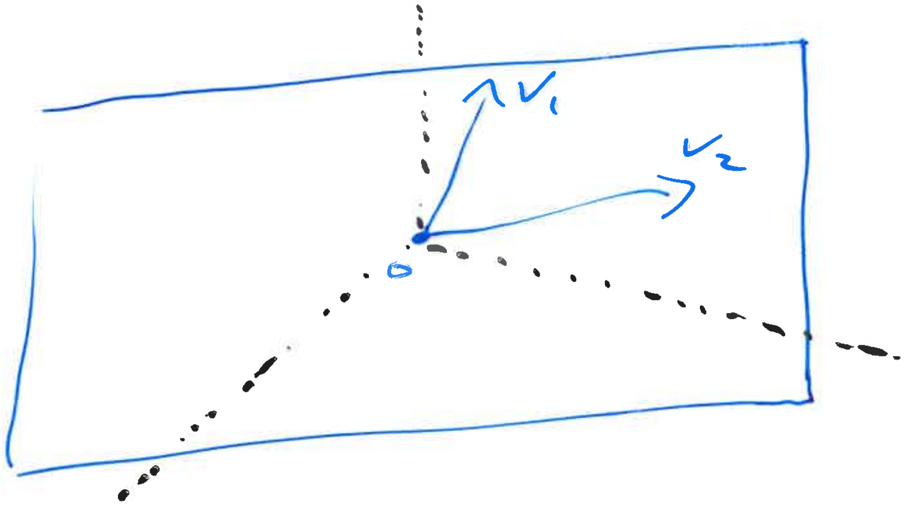
The span of two (linearly independent) vectors V_1 & V_2 is the plane through the origin that they lie in.



Ex: Prove that the vectors $V_1 = (1,-1,0)$, $V_2 = (1,0,2)$ & $V_3 = (0,1,1)$ are linearly independent.

i.e, Show that the only solution to

$$x_1 V_1 + x_2 V_2 + x_3 V_3 = (0,0,0)$$

is the trivial solution, $x_1 = x_2 = x_3 = 0$.

$\Rightarrow$ Solve

$$\begin{aligned} x_1 + x_2 \quad &= 0 \\ -x_1 \quad + x_3 &= 0 \\ 2x_2 + x_3 &= 0 \end{aligned}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right) \xrightarrow{R_2 \mapsto R_2 + R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 \mapsto R_3 - 2R_2]{R_1 \mapsto R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{R_2 \mapsto R_2 + R_3 \\ R_3 \mapsto -R_3}]{R_1 \mapsto R_1 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array}$$

Geometrically: The vectors v_1, v_2, v_3 don't lie in a plane, but point in 3 different "dimensions".

So we can use these 3 vectors $v_1 = (1,-1,0)$, $v_2 = (1,0,2)$ & $v_3 = (0,1,1)$ as axes in $\mathbb{R}^3$ instead of $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ & $e_3 = (0,0,1)$.

e.g., $v = (5,-1,7) = 2(1,-1,0) + 3(1,0,2) + 1(0,1,1)$.

In this case v is uniquely written as a linear combination of v_1, v_2 & v_3, i.e., there is a unique solution to

$$v = x_1 v_1 + x_2 v_2 + x_3 v_3.$$

Check by solving the system

$$\begin{aligned} x_1 + x_2 &= 5 \\ -x_1 + x_3 &= -1 \\ 2x_2 + x_3 &= 7 \end{aligned}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ -1 & 0 & 1 & -1 \\ 0 & 2 & 1 & 7 \end{array}\right) \xrightarrow{R_2 \mapsto R_2 + R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & 1 & 4 \\ 0 & 2 & 1 & 7 \end{array}\right)$$

$$\xrightarrow[R_3 \mapsto R_3 - 2R_2]{R_1 \mapsto R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -1 & -1 \end{array}\right)$$

$$\xrightarrow[\substack{R_2 \mapsto R_2 + R_3 \\ R_3 \mapsto -R_3}]{R_1 \mapsto R_1 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right) \Rightarrow \begin{aligned} x_1 &= 2 \\ x_2 &= 3 \\ x_3 &= 1 \end{aligned}$$

(Note that we repeated the same row operations)

Another way to say this is that every vector $w \in \mathbb{R}^3$ is in the span of $v_1 = (1,-1,0)$, $v_2 = (1,0,2)$ & $v_3 = (0,1,1)$.
i.e., $\mathbb{R}^3 = \text{Span}\{v_1, v_2, v_3\}$.

Recall also that v_1, v_2 & v_3 are linearly independent.

Any collection of 3 vectors $\{\omega_1, \omega_2, \omega_3\}$ such that

(i) $\mathbb{R}^3 = \text{Span}\{\omega_1, \omega_2, \omega_3\}$ &

(ii) $\{\omega_1, \omega_2, \omega_3\}$ are linearly independent.

is called a <u>basis</u> for $\mathbb{R}^3$. (A set of axes geometrically for $\mathbb{R}^3$).

<u>Ex:</u> Prove that $(1,2,1)$, $(1,3,1)$ & $(1,0,1)$ is not a basis.

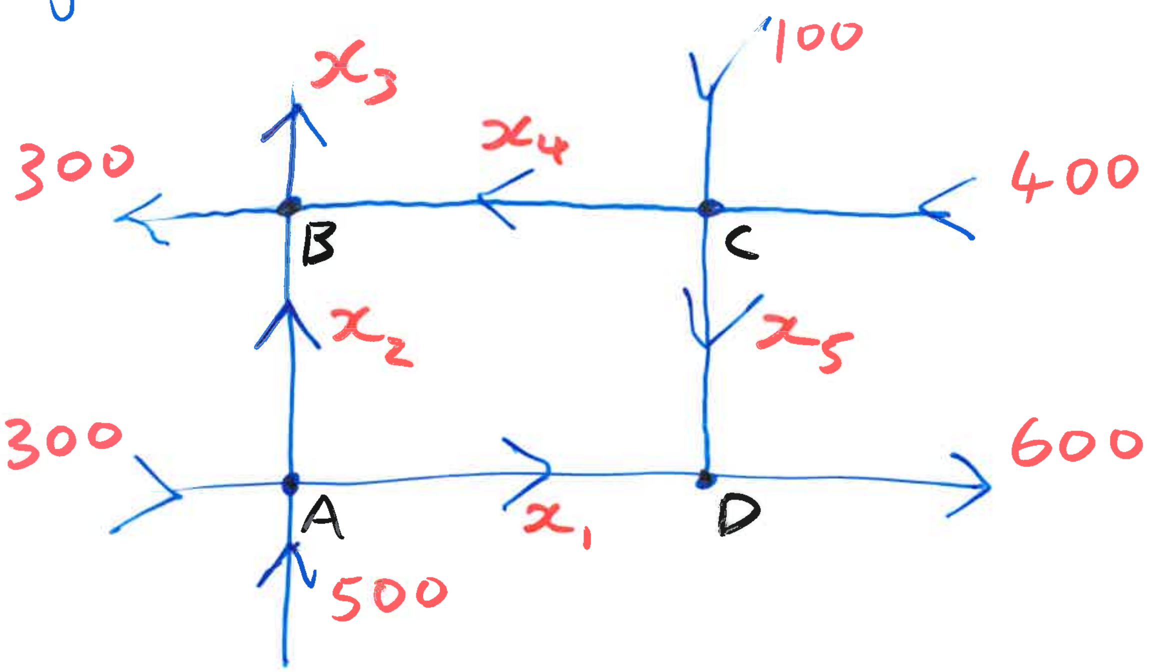
Hint: Can you express $(1,1,-2)$ as a linear combination of $(1,2,1)$, $(1,3,1)$ & $(1,0,1)$?

Applications:

Systems of linear eqns arise naturally in many areas of science, business, & maths & computing.

e.g., balancing chemical eqns or studying network flow.

Ex: Consider the traffic flow in a city, illustrated by the following diagram.



At each junction A, B, C, D, flow in = = flow out.

	Flow in		Flow out
A	$300+500$	$=$	x_1+x_2
B	x_2+x_4	$=$	$300+x_3$
C	$100+400$	$=$	x_4+x_5
D	x_1+x_5	$=$	600
Total:	$100+400+500+300$	$=$	$x_3+600+300$

Rearranging each eqn so it is of the form $\pm x_1 \pm x_2 \pm x_3 \pm x_4 \pm x_5 = n$ the augmented matrix is

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 0 & 1 & 0 & 0 & 400 \end{array}\right)$$

Exercise: Check that by putting this matrix in reduced echelon form that

- x_5 is free
- $x_1 = 600 - x_5$
- $x_2 = 200 + x_5$
- $x_3 = 400$
- $x_4 = 500 - x_5$

$$\Rightarrow S = \{(600-t, 200+t, 400, 500-t, t) \mid t \in \mathbb{R}\}$$

where $0 \leq x_5 \leq 500$. (Why?) (really $\mathbb{N}$)