Linear Independence of Vectors in $\mathbb{R}^n$

We have seen examples of systems of linear eqns which contain redundant equations in the sense that some of the equations may be obtained from the others i.e, in the sense that some of the equations can be written as a linear combination of the others.

Q: Given a collection of vectors in $\mathbb{R}^n$, can we write some of them as a linear combination of the others?

If not, we say that the collection is <u>linearly independent</u>. Otherwise, linearly dependent.

<u>Ex</u>: Is the collection $v_1 = (1,2,1)$, $v_2 = (3,-1,2)$
$v_3 = (9,4,7)$ linearly independent?

<u>Ans</u>: No, they are linearly dependent, as

$$(9,4,7) = 3(1,2,1) + 2(3,-1,2).$$

equivalently:

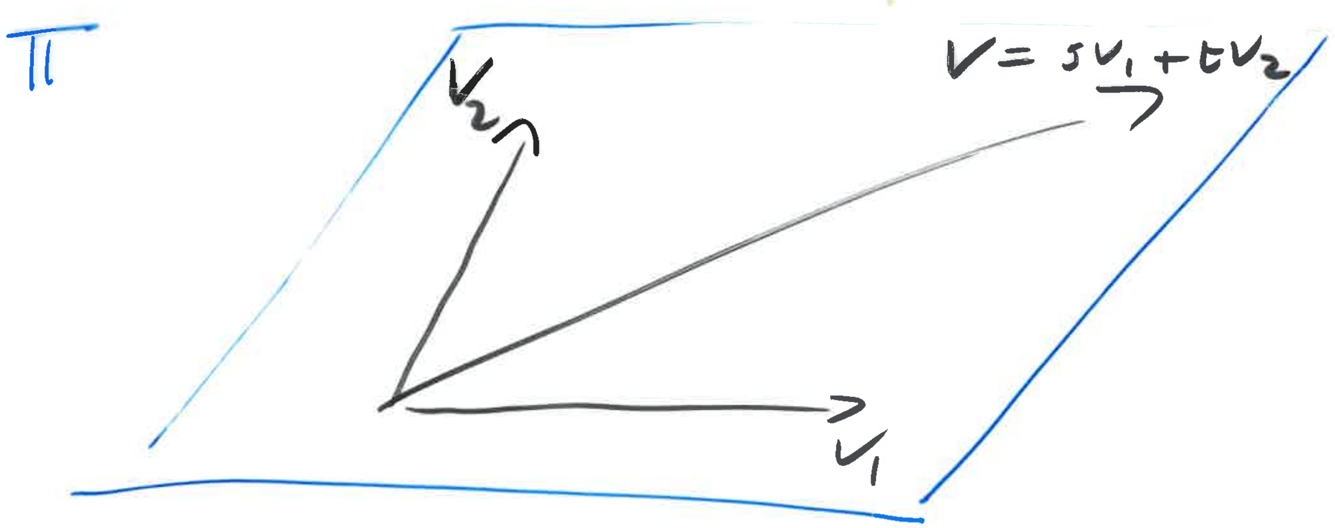
$$(9,4,7) - 3(1,2,1) - 2(3,-1,2) = (0,0,0).$$

i.e., $\exists\ x_1, x_2, x_3 \in \mathbb{R}$ not all zero, such that

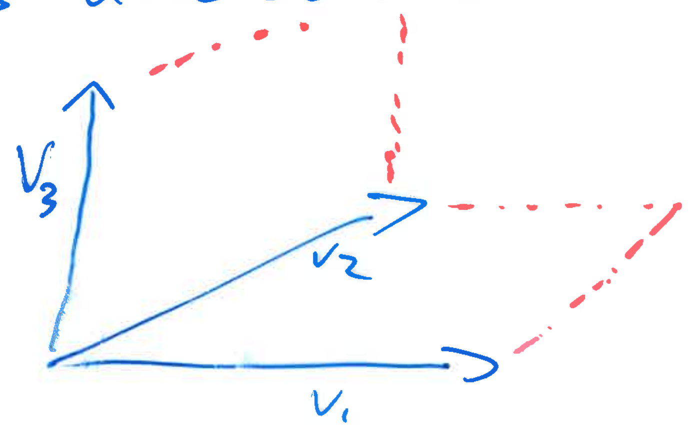
$$x_1(9,4,7) + x_2(1,2,1) + x_3(3,-1,2) = (0,0,0).$$

In this case, $x_1 = 1$, $x_2 = -3$, $x_3 = -2$.

Geometrically: $V_3 = (9,4,7)$ lies in the plane $\Pi := \{ v \in \mathbb{R}^3 \mid v = s v_1 + t v_2,\ s,t \in \mathbb{R} \}$.



If V_1, V_2 & V_3 were linearly independent they would not lie in a plane & we would need 3 dimensions to draw them. e.g.



(They form the sides of a "box")

Formal Definition: A collection of vectors $\{v_1, \dots, v_p\}$ in $\mathbb{R}^n$ is said to be <u>linearly independent</u> if, for $x_1, x_2, \dots, x_p \in \mathbb{R}$,

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0 = (0, 0, \dots, 0)$$

has only the trivial solution

$$x_1 = x_2 = \cdots = x_p = 0.$$

And the collection $\{v_1, \dots, v_p\}$ of vectors in $\mathbb{R}^n$ is said to be <u>linearly dependent</u> if $\exists$ real numbers $c_1, c_2, \dots, c_p$ not all zero such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0 = (0, 0, \dots, 0).$$

i.e., the vector equation $x_1 v_1 + \cdots + x_p v_p = 0$ has a non-trivial solution, $x_1 = c_1$, $x_2 = c_2$, etc.

Ex: Let $v_1 = (1, 2, 3)$, $v_2 = (4, 5, 6)$, $v_3 = (2, 1, 0)$.

Determine if the set $\{v_1, v_2, v_3\}$ is linearly independent.

i.e., Can we find x_1, x_2, x_3 not all zero such that

$$x_1 (1, 2, 3) + x_2 (4, 5, 6) + x_3 (2, 1, 0) = (0, 0, 0)$$

$$\Rightarrow \left.\begin{aligned} x_1 + 4x_2 + 2x_3 &= 0 \\ 2x_1 + 5x_2 + x_3 &= 0 \\ 3x_1 + 6x_2 &= 0. \end{aligned}\right\} A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right) \xrightarrow[R_3 \mapsto R_3 - 3R_1]{R_2 \mapsto R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 \mapsto R_3 - 2R_2} \left(\begin{array}{ccc|c} \boxed{1} & 4 & 2 & 0 \\ 0 & \boxed{-3} & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

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So x_1 & x_2 are non-free variables & x_3 is free to take any real number $t \in \mathbb{R}$ as its value. Then x_1 & x_2 are determined by x_3.

$$\Rightarrow \quad -3x_2 - 3x_3 = 0 \Rightarrow x_2 = -x_3 = -t$$

$$\& \quad x_1 + 4x_2 + 2x_3 = 0 \Rightarrow x_1 + 4(-t) + 2t = 0$$

$$\Rightarrow x_1 = 2t.$$

So all solutions are $\{(2t, -t, t) \in \mathbb{R}^3 \mid t \in \mathbb{R}\}$.

In particular $t \neq 0$, e.g., $t = 1$ gives

$$x_1 = 2, \quad x_2 = -1, \quad x_3 = 1$$

So: $2v_1 - v_2 + v_3 = 0$ (check)

Thus v_1, v_2 & v_3 are <u>not</u> linearly independent as we have found x_1, x_2, x_3 not all zero such that $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$

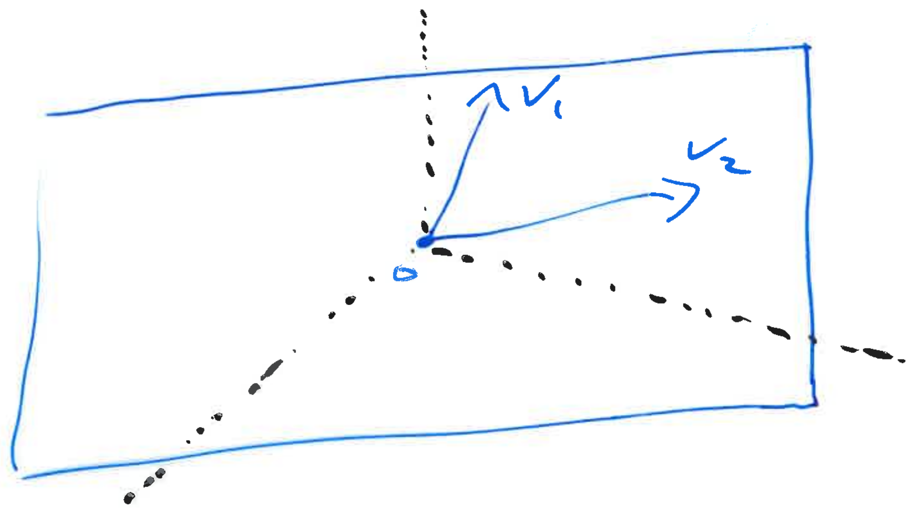
Alternatively: $v_3 = -2v_1 + v_2$, so v_3 lies in the "span" of v_1 & v_2.

<u>Definition</u>: If $v_1, \ldots, v_p$ are vectors in $\mathbb{R}^n$, the set of all linear combinations of $v_1, \ldots, v_p$, i.e., $\{v = c_1 v_1 + \cdots + c_p v_p \mid c_1, \ldots, c_p \in \mathbb{R}\}$ is called the subset of $\mathbb{R}^n$ <u>spanned</u> by (or generated by) $v_1, \ldots, v_n$. This is denoted by: $\text{Span}\{v_1, \ldots, v_p\}$ or $\langle v_1, \ldots, v_p \rangle$.

i.e., a vector $w \in \mathbb{R}^n$ is in the span of $v_1, \ldots, v_p$ if there exists a solution of the vector eqn $x_1 v_1 + \cdots + x_p v_p = w$.

Geometrically: The span of one vector v, $\text{Span}\{v\} = \{tv \mid t \in \mathbb{R}\}$, is the line through the origin in the direction v.

The span of two (linearly independent) vectors v_1 & v_2 is the plane through the origin that they lie in.



Ex: Prove that the vectors $v_1 = (1,-1,0)$, $v_2 = (1,0,2)$ & $v_3 = (0,1,1)$ are linearly independent.

i.e. Show that the only solution to

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = (0,0,0)$$

is the trivial solution, $x_1 = x_2 = x_3 = 0$.

$\Rightarrow$ Solve

$$\begin{aligned} x_1 + x_2 &= 0 \\ -x_1 + x_3 &= 0 \\ 2x_2 + x_3 &= 0 \end{aligned}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right) \xrightarrow{R_2 \mapsto R_2 + R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right)$$