Note! The set of all solutions S can also be written as

$$S = \{(-6s-3t, s, 4t, t, 0) + (0,0,5,0,7) \in \mathbb{R}^5 \mid s,t \in \mathbb{R}\}$$

and (check!)

$$S' = \{(-6s-3t, s, 4t, t, 0) \in \mathbb{R}^5 \mid s,t \in \mathbb{R}\}$$

is the set of solutions to the system

$$Ax = 0, \quad A = \begin{pmatrix} 1 & 6 & 2 & -5 & -2 \\ 0 & 0 & 2 & -8 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

called the "corresponding homogeneous system" & $p = (0,0,5,0,7)$ is one particular solution of $Ax = b$.

This is a general fact for systems of (consistent) linear eqns, i.e, the solutions of $Ax = b$ are all of the form $X' + P$ where X' is a solution of $Ax = 0$ and P is one "particular" solution of $Ax = b$.

Ex: Solve the Homogeneous system of eqns

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 0 \\ 2x_1 - 3x_2 + 4x_3 - 3x_4 &= 0 \\ 3x_1 - 5x_2 + 5x_3 - 4x_4 &= 0 \\ -x_1 + x_2 - 3x_3 + 2x_4 &= 0 \end{aligned}$$

The augmented matrix is:

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 2 & -3 & 4 & -3 & 0 \\ 3 & -5 & 5 & -4 & 0 \\ -1 & 1 & -3 & 2 & 0 \end{array}\right) \xrightarrow[\substack{R_3 \mapsto R_3 - 3R_1 \\ R_4 \mapsto R_4 + R_1}]{R_2 \mapsto R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R_4 \mapsto R_4 + R_2]{R_3 \mapsto R_3 - R_2} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$ (Echelon form)

$$\xrightarrow{R_1 \mapsto R_1 + 2R_2} \left(\begin{array}{cccc|c} 1 & 0 & 5 & -3 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$ (Reduced Echelon form.)

So the columns with a (non-zero) pivot are Col 1 & Col 2, so x_1 & x_2 are non-free, and x_3 & x_4 are free variables. i.e, they can be any real numbers r & s.

So the general solution (or set of all sol'ns) is

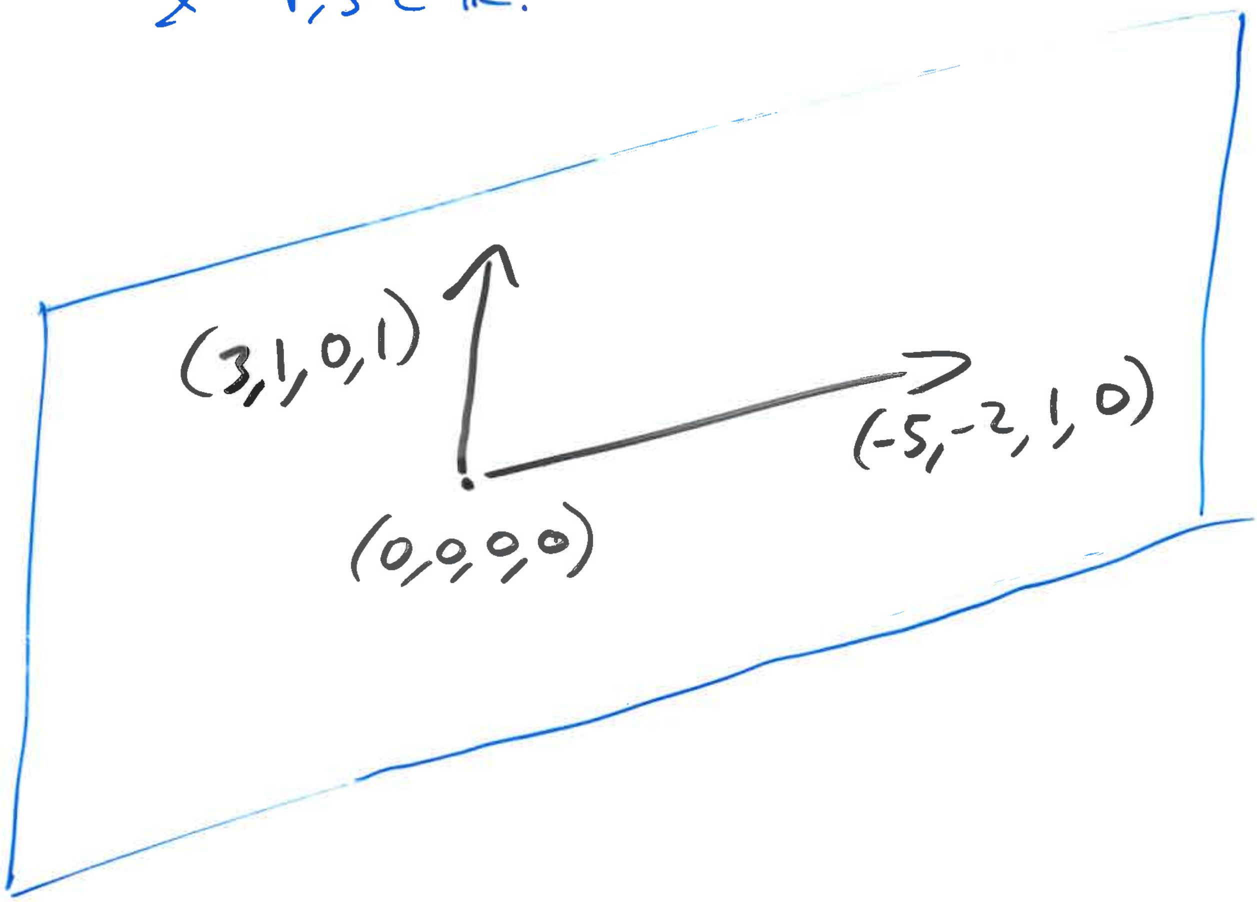
$$\begin{aligned} x_1 &= -5x_3 + 3x_4 \\ x_2 &= -2x_3 + x_4 \\ x_3 &= r \\ x_4 &= s. \end{aligned}$$

i.e., $S = \{(-5r+3s, -2r+s, r, s) \in \mathbb{R}^4 \mid r, s \in \mathbb{R}\}$.

Geometrically, S is a 2-dimensional plane in $\mathbb{R}^4$ through the origin $(0,0,0,0)$ because if $x \in S$ then $x = (-5r+3s, -2r+s, r, s)$

$$= r(-5,-2,1,0) + s(3,1,0,1)$$

& $r, s \in \mathbb{R}$.



Note that letting $r = s = 0$ we see that $(0,0,0,0) \in S$.