Definition: We say a system of linear equations is **consistent** if it has a solution, and otherwise it is **inconsistent**.

Note: A consistent system can have infinitely many solutions, or a unique solution.

Ex) Our last example had a unique solution, so is consistent.

Ex) Find the solutions (if any) of

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$

Ans: $(A|b) = \left(\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right)$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} \boxed{2} & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \mapsto R_3 - \frac{5}{2}R_1} \left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & \boxed{1} & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array}\right)$$

$$\underline{R_3 \mapsto R_3 + \tfrac{1}{2} R_2} \quad \left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{array}\right)$$

But Row 3 now says that $0x_1 + 0x_2 + 0x_3 = \frac{5}{2}$

This is a clear contradiction.

So there are no solutions, this system is inconsistent.

Echelon form & Reduced Echelon form

Recall in our earlier example, solving the system of linear equations with augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 3 & 2 & 100 \\ 1 & 1 & 4 & 70 \\ 20 & 10 & 10 & 500 \end{array}\right).$$

We used elementary row operations to put it in Echelon form:

$$\left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & 1 & -6 & -40 \\ 0 & 0 & -130 & -1300 \end{array}\right)$$

and used "back - substitution" to get

$x_1 = 10$, $x_2 = 20$, $x_3 = 10$.

Alternatively, we could have continued and placed as many zeros as possible above the pivots in each column, to put the matrix in Reduced Echelon form:

$$\left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & 1 & -6 & -40 \\ 0 & 0 & -130 & -1300 \end{array}\right) \xrightarrow{R_3 \mapsto -\frac{1}{130} R_3} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & \boxed{1} & -6 & -40 \\ 0 & 0 & 1 & 10 \end{array}\right)$$

$$\xrightarrow{R_1 \mapsto R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 10 & 110 \\ 0 & 1 & -6 & -40 \\ 0 & 0 & \boxed{1} & 10 \end{array}\right)$$

$$\xrightarrow[R_2 \mapsto R_2 + 6R_3]{R_1 \mapsto R_1 - 10R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 10 \end{array}\right)$$

$$\Rightarrow x_1 = 10, \quad x_2 = 20, \quad x_3 = 10.$$

The reduced Echelon form is useful for looking for all solutions, i.e., the "general solution" of a system with infinitely many solutions.

Ex Find the general solution of the system whose augmented matrix has been reduced to:

$$\left(\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right)$$

which is already in Echelon form.
Now put it in reduced Echelon form.

$$\left(\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right) \xrightarrow[R_1 \mapsto R_1 + 2R_3]{R_2 \mapsto R_2 + R_3} \left(\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right)$$

$$\xrightarrow[R_2 \mapsto \frac{1}{2} R_2]{R_1 \mapsto R_1 - R_2} \left(\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right)$$

So
$$x_1 + 6x_2 + 3x_4 = 0$$
$$x_3 - 4x_4 = 5$$
$$x_5 = 7$$

The columns with a pivot (a first non-zero in a row) are col 1, col 3, col 5.
So x_1, x_3 & x_5 are "non-free variables".
The remaining variables x_2 & x_4 are called "free variables"

Free variables are so-called because they are free to be any real numbers, and the non-free variable are determined in terms of the free ones.

So
$$x_1 = -6x_2 - 3x_4$$
$$x_3 = 5 + 4x_4$$
$$x_5 = 7$$

where x_2 is free, i.e., $x_2 = s \in \mathbb{R}$ (any s).

x_4 is free, i.e., $x_4 = t \in \mathbb{R}$ (any t).

So the "solution space" is, the set of all solutions, is
$$S := \{(-6s-3t, s, 5+4t, t, 7) \in \mathbb{R}^5 \mid s, t \in \mathbb{R}\}$$

Ex: Take special values for t and s & verify that

e.g., $s=3$, $t=2$

e.g., $(-24, 3, 13, 2, 7)$ does satisfy the three original equations.

Or $s=0$, $t=0$ so that $(0,0,5,0,7)$ is a solution of the original $Ax=b$.