Solving Systems of Linear Equations and Echelon Form

Some systems are easily solved, e.g.,

(i) $x_1 + 2x_2 + x_3 = 16$

(ii) $x_2 + 2x_3 = 10$

(iii) $3x_3 = 9$

This is because of the "upper diagonal shape of the equations. i.e., eq (ii) has one less unknown than eq (i), & eq (iii) has one less than eq (ii).

Soln: A solution to the above system of linear eq's (i.e., multiple equations with unknowns x_1, x_2, x_3 which are multiplied by real numbers and added) is a vector of the form $c = (c_1, c_2, c_3)$ so that if $x_1 = c_1$, $x_2 = c_2$, & $x_3 = c_3$ then all eq's (i), (ii) & (iii) are satisfied.

In this case: (iii) $\Rightarrow x_3 = 3$.

Back substituting this value into (ii)

$\Rightarrow x_2 + 2(3) = 10 \quad \Rightarrow x_2 = 4$.

Back substituting both into (i) then

$\Rightarrow x_1 + 2x_2 + x_3 = x_1 + 2(4) + 3 = 16$

$\Rightarrow x_1 = 5$.

So in this case we have a unique solution, $x_1 = 5$, $x_2 = 4$, $x_3 = 3$.
or simply $(5, 4, 3)$.

We will use the definition of matrix multiplication to write a system of linear equations as one matrix equation.

Using the example above, let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \& \quad b = \begin{pmatrix} 16 \\ 10 \\ 9 \end{pmatrix}$$

Then our system can be written as :

$$Ax = b.$$

IDEA: Change a given system of linear equations to a system that has a "simple" form, as in our example, and that has the same solutions. i.e., change to a so called "equivalent system".

<u>Question:</u> What can we do to a system of linear equations without changing the solutions?

Answer: <u>Elementary Operations</u>

(1) Interchange 2 equations.

(2) Multiply an equation by a non-zero real number.

(3) Replace an equation by itself plus a multiple (non-zero) of some other equation in the system.

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we write the system of equations in "Extended matrix notation" i.e.,

(i) $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

(ii) $a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$

$\vdots$

(m) $a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$

we rewrite as $Ax = b$.

Or simply encode this info in the extended (or augmented) matrix

$$(A|b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} & b_m \end{array}\right)$$

So that equation:

(i) corresponds to Row 1 of $(A|b)$

(ii) " " " 2 " "

$\vdots$

(m) " " " m " "

Letting R_i denote row i of $(A|b)$, then the elementary (row) operations are written as:

(1) Interchange R_i & R_j

(2) Replace R_i by kR_i, $k \neq 0$, $k \in \mathbb{R}$.

(3) Replace R_i by $R_i + kR_j$, $j \neq i$, $k \in \mathbb{R}$.

<u>Ex</u> Consider the system of 3 linear equations in 3 unknowns x_1, x_2 & x_3.

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 100 \\ x_1 + x_2 + 4x_3 &= 70 \\ 20x_1 + 10x_2 + 10x_3 &= 500 \end{aligned} \Longleftrightarrow \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 4 \\ 20 & 10 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 100 \\ 70 \\ 500 \end{pmatrix}$$

Keep track in the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 3 & 2 & 100 \\ 1 & 1 & 4 & 70 \\ 20 & 10 & 10 & 500 \end{array}\right) := (A|b)$$

Now use the elementary row operations so that $(A|b)$ is in so called "Echelon form", i.e., "upper triangular form" like the first "simple" eqs we solved before.

The idea is to use the first non-zero entry in a row (Pivot) to place zeros in the column entries below it.

Pivot

$$\left(\begin{array}{ccc|c} 2 & 3 & 2 & 100 \\ 1 & 1 & 4 & 70 \\ 20 & 10 & 10 & 500 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} \boxed{1} & 1 & 4 & 70 \\ 2 & 3 & 2 & 100 \\ 20 & 10 & 10 & 500 \end{array}\right)$$

Pivot

$$\xrightarrow[R_3 \mapsto R_3 - 20R_1]{R_2 \mapsto R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & \boxed{1} & -6 & -40 \\ 0 & -10 & -70 & -900 \end{array}\right)$$

$$\xrightarrow{R_3 \mapsto R_3 + 10R_2} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & 1 & -6 & -40 \\ 0 & 0 & -130 & -1300 \end{array}\right)$$

0's below diagonal.

So: Row 3 now says $-130x_3 = -1300$

$\Rightarrow x_3 = 10$

Row 2 says $x_2 - 6x_3 = -40$

$\Rightarrow x_2 - 60 = -40$

$\Rightarrow x_2 = 20.$

Row 1 says $x_1 + x_2 + 4x_3 = 70$

$\Rightarrow x_1 + 20 + 40 = 70$

$\Rightarrow x_1 = 10.$