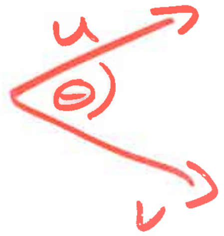
Properties of the dot product:

(i) $u \cdot v = v \cdot u \quad \forall\, u, v \in \mathbb{R}^n$

(ii) $u \cdot (v + w) = u \cdot v + u \cdot w \quad \forall\, u, v, w \in \mathbb{R}^n$

(iii) $(ru) \cdot v = u \cdot (rv) = r(u \cdot v) \quad \forall\, u, v \in \mathbb{R}^n,\ r \in \mathbb{R}$

(iv) In $\mathbb{R}^2$, $u \cdot v = \|u\| \|v\| \cos\theta$, where θ is the angle between u & v



As in higher dimensions, any 2 vectors define a 2-dimensional plane, we also have the same notion of angle θ.

Corollary: $u \cdot v = 0 \iff \cos\theta = 0$

i.e., $\theta = \frac{\pi}{2}$

So u & v are perpendicular or orthogonal.

Def^n: The hyperplane in $\mathbb{R}^n$ with normal $n = (n_1, \dots, n_n)$ through the origin $O = (0, 0, \dots, 0)$ is the set of vectors (or points) $x = (x_1, \dots, x_n)$ that are perpendicular to n.

So $\underset{\text{normal}}{\underline{n}} \cdot x = n_1 x_1 + n_2 x_2 + \dots + n_n x_n = 0.$

When $\underset{\text{dimension}}{\underline{n}} = 3$ we just call it the plane.

Ex, If the plane goes through some other point, say $p = (p_1, p_2, p_3)$, and again has normal $n = (n_1, n_2, n_3) \neq 0$.

Then the equation of the plane is,

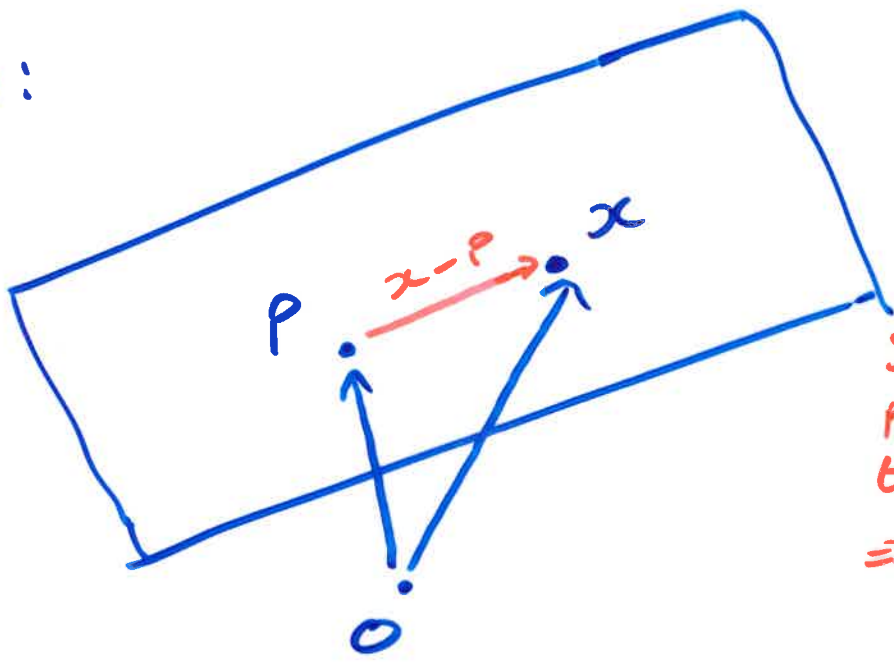
$$\Pi : \{ (x_1, x_2, x_3) = x \mid (x - p) \cdot n = 0 \}.$$

So the eqn is $x \cdot n - p \cdot n = 0$

i.e., $x \cdot n = p \cdot n = d \in \mathbb{R}$

$\Rightarrow x_1 n_1 + x_2 n_2 + x_3 n_3 = d.$

Picture :



See $x-p$ is a vector "on the plane".

So $x-p$ is perpendicular to the normal.

$\Rightarrow (x-p) \cdot n = 0.$

Ex) Find the equation of the plane in $\mathbb{R}^3$ with normal $n = (2,3,1)$ passing through the point $p = (1,1,2)$.

If $x = (x_1, x_2, x_3)$ lies on the plane, then

$$n \cdot x = n \cdot p \Rightarrow (2,3,1) \cdot (x_1, x_2, x_3) = (2,3,1) \cdot (1,1,2)$$

$$\Rightarrow 2x_1 + 3x_2 + x_3 = 7.$$

CHECK that $p = (1,1,2)$ lies on this plane:

$$2(1) + 3(1) + 2 = 7 \checkmark$$

It is intuitively clear that there is a unique plane π through 3 points (not all on the same line) in $\mathbb{R}^3$.

How do we find the equation of π?

We need to solve 3 linear eqn's in the 3 unknowns n_1, n_2 & n_3

The eqn is of the form

$$n_1 x_1 + n_2 x_2 + n_3 x_3 = d \quad (\text{some } d \in \mathbb{R}).$$

$\Rightarrow$ Suppose 3 pts are $(1,1,2)$, $(-3,4,6)$ & $(0,5,7)$.

Then

$$\left.\begin{array}{ll} \text{(i)} & n_1 + n_2 + 2n_3 = d_1 \\ \text{(ii)} & -3n_1 + 4n_2 + 6n_3 = d_2 \\ \text{(iii)} & 5n_2 + 7n_3 = d_3 \end{array}\right\} \text{The } d_i\text{'s are given real numbers.}$$

We expect this to have a solution as we have 3 equations and 3 unknowns.

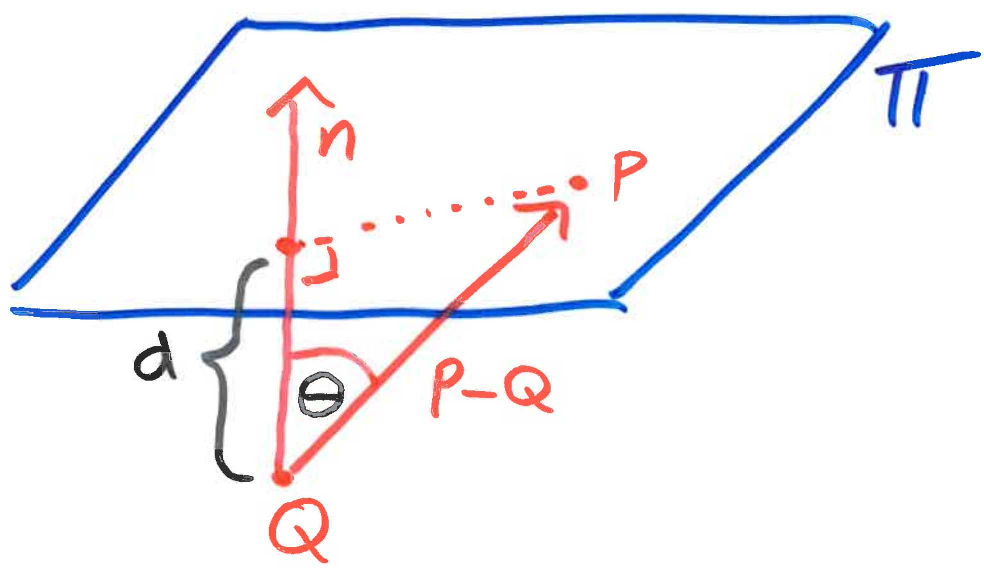
In $\mathbb{R}^2$, we expect 2 lines to intersect in exactly one point.

In $\mathbb{R}^3$ we expect 3 planes to intersect in one point (not always!).

Another application of the dot product

Find the distance from a point Q to the plane π with normal n & passing through a point $P \in \mathbb{R}^3$.

We assume that $Q \notin \pi$.



$d = \|P-Q\| \cos\theta$ (from picture).

$\|P-Q\|\,\|n\| \cos\theta = (P-Q)\cdot n$

Combining $\Longrightarrow$ $d = \dfrac{|(P-Q)\cdot n|}{\|n\|}$