Row Operations and Determinants

Theorem: Let A be a square matrix.

(a) If a multiple of one row of A is added to another row of A to produce a matrix B then:

$$\det B = \det A.$$

(b) If two rows of A are interchanged to produce a matrix B, then:

$$\det B = -\det A.$$

(c) If a row of A is multiplied by $k \in \mathbb{R}$, $k \neq 0$, to produce a matrix B, then:

$$\det B = k \cdot \det A.$$

Ex: Let $A = \begin{pmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{pmatrix}$

Simplify with row operations, and track the determinant

$$\underset{A}{\begin{pmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{pmatrix}} \xrightarrow[R_3 \mapsto R_3 + R_1]{R_2 \mapsto R_2 + 2R_1} \underset{B}{\begin{pmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{pmatrix}}$$

By Theorem (a): $\det B = \det A$.

$$\underset{B}{\begin{pmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{pmatrix}} \xrightarrow{R_2 \leftrightarrow R_3} \underset{C}{\begin{pmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{pmatrix}}$$

By Theorem (b): $\det C = -\det B = -\det A$.

Now: when a matrix is "upper-triangular", i.e., all entries below the main diagonal are zero, then the determinant is the product of the entries on the main diagonal.

(Exercise: Convince yourself this is true, using the cofactor method for calculating determinants.).

Thus $\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5)$

$= 15.$

Ex $\begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}$

by Thm (c).

$\overset{(*)}{=} 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix} \overset{(*)}{=} 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix}$

$\overset{(*)}{=} 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & 2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2(1)(3)(-6)(1)$

$= -36$

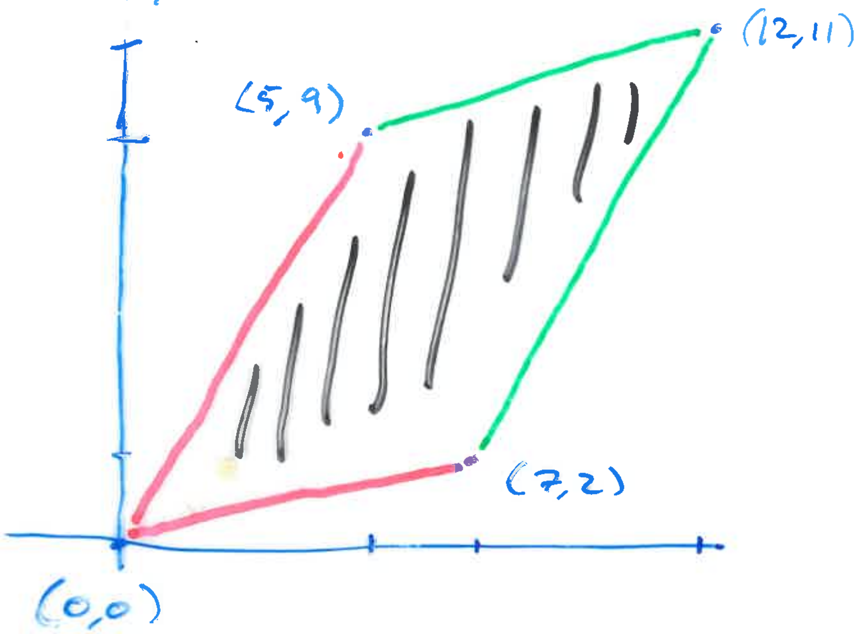
(*) All by Thm (a).

Exercise: Determine what the row operations used in each step above were.

A Geometric Explanation for the behaviour of the determinant with respect to Elementary row operations.

Recall that in 2 dimensional space, the area of a parallelogram determined by two vectors (a,b) and (c,d) is given by $|\det A|$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Ex, Let $(a,b) = (5,9)$ and $(c,d) = (7,2)$.



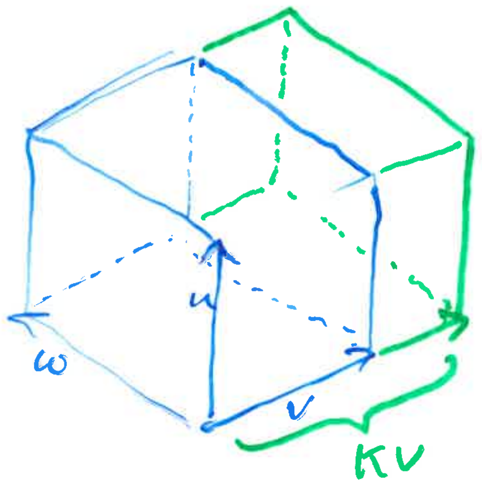
Then the parallelogram with vertices $(0,0)$, $(5,9)$, $(7,2)$ and $(5,9)+(7,2) = (12,11)$ is $|(5)(2)-(9)(7)| = |-53| = 53$.

Exercise: Verify this by calculating the area above some other way.

Recall also that in 3 dimensions, the volume of a box with one vertex at $(0,0,0)$ and three sides corresponding to vectors, is given by $|\det A|$ where A is the 3×3 matrix with these 3 vectors as rows.

This remains true in n-dimensional space.

So, for example, in $\mathbb{R}^3$ if we multiply a row by some $k \in \mathbb{R}$, $k \neq 0$, we scale the volume of the box with sides the rows of A by a factor of k.



(*) Note: If $k=0$ the box collapses to a rectangle and then the volume is zero.

This explains part (c) of the previous Theorem.

(*) More generally, if a matrix in n-dimensions has two rows in the same dimension, the box has 2 sides collapsed into one. So it is now an $(n-1)$-dimensional box and has n-volume $=0$, i.e, $\det A = 0$.

The following explains part (a) of the previous theorem.

Let $A = \begin{pmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \\ R_n \end{pmatrix}$ and $B = \begin{pmatrix} R_1 \\ \vdots \\ R_i + kR_j \\ \vdots \\ R_j \\ \vdots \\ R_n \end{pmatrix}$

Then $\det B = \det \begin{pmatrix} R_1 \\ \vdots \\ R_i + kR_j \\ \vdots \\ R_j \\ \vdots \\ R_n \end{pmatrix} = \det \begin{pmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \\ R_n \end{pmatrix}$

$+ k \det \begin{pmatrix} R_1 \\ \vdots \\ R_j \\ R_j \\ \vdots \\ R_n \end{pmatrix} = 0$

Two rows in same direction.

$\Rightarrow \det B = \det A + 0.$

Picture in $\mathbb{R}^2$