Inverses via determinants.

Recall: For a 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

det A, or $|A|$ or $\Delta := a_{11}a_{22} - a_{12}a_{21}$

If $|A| \neq 0$ then $A^{-1} = \dfrac{1}{|A|}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$

and if $|A| = 0$, A^{-1} does not exist.

Now: For a 3×3 matrix A, its determinant det A, or $|A|$ is defined or given in terms of determinants of 2×2 submatrices. That is:

$$|A| = a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}|$$

where: (a_{11}, a_{12}, a_{13}) is the first row of A

and: A_{ij} is the 2×2 matrix obtained from A by deleting row i and column j.

Ex, Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Then $A_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}$ $A_{12} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}$, $A_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$

$$\Rightarrow |A| = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(45-48) - 2(36-42) + 3(32-35)$$

$$= 0$$

We can similary inductively define the determinant of an $n \times n$ matrix A in terms of determinants of $(n-1)\times(n-1)$ submatrices. Thus by repeating, we eventually calculate several determinants of 2×2 matrices.

For example, if A is a 4×4 matrix. we define

$$|A| = a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}| - a_{14}|A_{14}|$$

where now the A_{ij} are 3×3.

In general:

Let A be an $n\times n$ matrix $(n\geqslant 2)$.

Then

$$|A| = a_{11}|A_{11}| - a_{12}|A_{12}| + \cdots + (-1)^{n+1} a_{1n}|A_{1n}|$$

$$= \sum_{i=1}^{n} (-1)^{i+1} a_{1i}|A_{1i}|.$$

(Be aware of the alternating sign $+ - + - \cdots$)

We often build the alternating sign into the notation as follows:

$$C_{ij} := (-1)^{i+j}|A_{ij}|$$

which is called the (i,j) cofactor.

Then $$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

$$= \sum_{k=1}^{n} a_{1k}C_{1k}$$

This is called "the cofactor expansion across the first row of A."

<u>Theorem</u>: The determinant of an $n \times n$ matrix A can be computed with a cofactor expansion across any row, or down any column.

The cofactor expansion across the i^{th} row is

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$$

& the cofactor expansion down the j^{th} column is

$$|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}.$$

(*)

Ex) $A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}$

We calculate the cofactor expansion along the third row of A:

$$|A| = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

(*)

where $C_{31} = (-1)^{3+1}|A_{31}| = +\begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix}$

$= -5$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -(-1) = 1$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = +(4-10) = -6.$$

$$\Rightarrow |A| = 0(-5) + (-2)(1) + 0(-6) = -2.$$

Remark: Observe that row 3 was a good choice because there were two zeros. i.e., $a_{31} = a_{33} = 0$. Thus we didn't really need to calculate C_{31} and C_{33}.

Now let's take the cofactor expansion down some column. Again, column 3 is a good choice as $a_{13} = a_{33} = 0$, so we need only calculate C_{23}.

So $$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} = -(-2) = 2.$$

Then $$|A| = 0C_{13} + (-1)C_{23} + 0C_{33}$$
$$= (-1)(2) = -2.$$ as before.

Exercise: Calculate the cofactor expansion along at least one other row and column of A.

Ex Compute $|A|$ where

$$A = \begin{pmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{pmatrix}$$

Use the cofactor expansion down the first column.

$$\Rightarrow |A| = 3 \cdot \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix}$$

- first column again

$$= 3 \cdot \left(2 \cdot \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} \right)$$

– third row

$$= 3 \cdot 2 \cdot \left(-2 \cdot (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \right)$$

$$= 3 \cdot 2 \cdot \big((-2)(-1)(-1) \big) = -12.$$