On the other hand, consider the matrix

$$A = \begin{pmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{pmatrix}$$

which we have seen has no inverse.

There are $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} := 0$

such that $AX = 0$, because

$$\left(\begin{array}{ccc|c} 0 & 3 & -5 & 0 \\ 1 & 0 & 2 & 0 \\ -4 & -9 & 7 & 0 \end{array}\right) \xrightarrow[\text{to}]{\text{Row reduces}} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

which has no pivot in Row 3, so x_3 is free, i.e., $x_3 = t \in \mathbb{R}$. Then

$$3x_2 = -5x_3 = -5t \Rightarrow x_2 = -\frac{5}{3}t.$$

$$\& \quad x_1 = -2x_3 = -2t \Rightarrow x_1 = -2t$$

So the solutions to $AX = 0$ are the vectors of the form $X = \begin{pmatrix} -2t \\ -5/3\, t \\ t \end{pmatrix}$

where $t \in \mathbb{R}$.

$$= t \begin{pmatrix} -2 \\ -5/3 \\ 1 \end{pmatrix}$$

Definition: The set of vectors X satisfying $AX = 0$ for a matrix A is called the kernel of A, denoted

$$\ker A := \{X \in \mathbb{R}^n \mid AX = 0\}.$$

In our last example, $\ker A = \operatorname{span}\left\{\begin{pmatrix}-2\\-5/3\\1\end{pmatrix}\right\}$.

i.e., the line in the direction of the vector $(-2, -5/3, 1)$ through the origin.

Note: the dimension of the kernel of A + rank A = 3.

"dim ker A."

Theorem: Let A be an $n \times n$ matrix.

Then $\dim \ker A + \operatorname{rank} A = n$.

Recall from earlier lectures the matrix

$$A = \begin{pmatrix} 1 & -2 & 1 & -1 \\ 2 & -3 & 4 & -3 \\ 3 & -5 & 5 & -4 \\ -1 & 1 & -3 & 2 \end{pmatrix} \xrightarrow{\text{"Row ops"}} \begin{pmatrix} 1 & 0 & 5 & -3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Clearly rows 1 & 2 are linearly independent $\Rightarrow$ rank $A = 2$.

Further, solving $AX = 0$, we see from above that x_3 & x_4 are free variables. So $x_3 = t$ and $x_4 = s$ and from rows 1 & 2 we get

$$x_2 = -2t - s$$
$$x_1 = -5t + 3s$$

$$\Rightarrow X = \begin{pmatrix} -5t+3s \\ -2t-s \\ t \\ s \end{pmatrix} = t\begin{pmatrix} -5 \\ -2 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

i.e., there is a 2-dimensional solution space to $AX = 0$, so $\dim \ker A = 2$.

So we see that rank $A + \dim \ker A = 4$.

Row operations via Matrix Multiplication

We observe that performing row operations on an $n \times n$ matrix A can be done as follows:

① Perform the same operation to the Identity matrix. Then

② Multiply A on the left by the matrix obtained in step ①.

In particular, if B is the matrix obtained in step ①, we calculate BA. (Not AB!).

<u>Ex:</u> Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 4 \end{pmatrix}$

To perform $R_1 \longmapsto R_1 + 2R_2$ for example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 7 \\ 2 & 0 & 3 \\ 1 & 2 & 4 \end{pmatrix}.$

e.g., $R_1 \leftrightarrow R_2$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

and then

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 2 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$$

So, it should be no surprise that when we perform row operations so that

$$(A \mid I) \longrightarrow (I \mid B)$$

Then $B = A^{-1}$. Why? The operations required to turn A into I, are equivalent to multiplying on the left by B. Hence $BA = I$, so $B = A^{-1}$.