$$= \begin{pmatrix} 2+9 & 4-3 & 2+6 \\ 1+3 & 2-1 & 1+2 \end{pmatrix} = \begin{pmatrix} 11 & 1 & 8 \\ 4 & 1 & 3 \end{pmatrix}$$

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## Finding the inverse of a matrix by row operations.

Recall that the inverse of an $n \times n$ matrix $A$ is another $n \times n$ matrix denoted by $A^{-1}$, such that

$$AA^{-1} = A^{-1}A = \underbrace{I_n}_{\text{The identity matrix.}} := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ & & & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

We don't know $A^{-1}$, so lets find its 3 columns $X_1, X_2, X_3$ as follows (here $A$ is $3 \times 3$).

$$A^{-1} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ X_1 & X_2 & X_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

So since $AA^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\Rightarrow AX_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad AX_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \& \quad AX_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{pmatrix}$.

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = X_1$ be the first column of $A^{-1}$.

Then $AX_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, so to find $X_1$ we solve the system.

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 2 & 5 & -3 & 0 \\ -3 & 2 & -4 & 0 \end{array}\right) \xrightarrow[\text{echelon form to get}]{\text{Put in reduced}}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array}\right).$$

Likewise, to find $X_2 = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ we solve $AX_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

i.e $\left(\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 2 & 5 & -3 & 1 \\ -3 & 2 & -4 & 0 \end{array}\right) \xrightarrow[\text{reduced echelon form.}]{\text{Put in}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & w \end{array}\right)$

But we just did the same set of row operations again! So: Do all three at the same time. i.e.,

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 5 & -3 & 0 & 1 & 0 \\ -3 & 2 & -4 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right).$$

So:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 5 & -3 & 0 & 1 & 0 \\ -3 & 2 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \longmapsto R_3+3R_1]{R_2 \longmapsto R_2-2R_1} \left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 11 & -10 & 3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \longmapsto R_3+11R_2]{R_1 \longmapsto R_1+3R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -5 & 3 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -19 & 11 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 \longmapsto -R_2+R_3]{R_1 \longmapsto R_1-R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 14 & -8 & -1 \\ -17 & 10 & 1 \\ -19 & 11 & 1 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right)$$

Check: $\begin{pmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{pmatrix}\begin{pmatrix} 14 & -8 & -1 \\ -17 & 10 & 1 \\ -19 & 11 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So what can go wrong so that $A^{-1}$ doesn't exist?

If A didn't have 3 pivot columns we wouldn't be able to reduce to $(I|A^{-1})$.

Ex, Let $A = \begin{pmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{pmatrix}$

Then $(A|I) = \left(\begin{array}{ccc|ccc} 0 & 3 & -5 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ -4 & -9 & 7 & 0 & 0 & 1 \end{array}\right)$

$$\xrightarrow{R_1 \Leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -5 & 1 & 0 & 0 \\ -4 & -9 & 7 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \mapsto R_3 + 4R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -5 & 1 & 0 & 0 \\ 0 & -9 & 15 & 0 & 4 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + 3R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 4 & 1 \end{array}\right)$$

No third pivot $\Rightarrow$ inconsistent equations.

Another way to think about the problem with $\begin{pmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{pmatrix}$ not having an inverse is that the rows are not linearly independent. We reduced the matrix to $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & -9 & 15 \end{pmatrix}$ so clearly row 3 = −3 row 2

i.e they are not linearly independent.

The number of linearly independent rows of a matrix doesn't change on performing row operations, but it makes it easier to spot.

e.g. $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & 0 \end{pmatrix}$ clearly has 2 linearly independent rows.

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Definition: Let $A$ be an $n \times n$ matrix. The Rank of $A$, denoted rank $A$, is the number of linearly independent rows of $A$.

(Aside: It can be shown that rank $A$ is also equal to the number of linearly independent columns of $A$.).

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Theorem: $A$ is an invertible matrix, i.e., $A^{-1}$ exists, if and only if rank $A = n$.

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Note: If $A^{-1}$ exists and $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Then $AX = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow X = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Why? Because

$$AX = 0 \Rightarrow A^{-1}AX = A^{-1}0 = 0$$

$$\Rightarrow I_n X = 0 \Rightarrow X = 0$$