Thus $L(v) = L((x,y))$ is obtained as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

i.e., $L(v) = (ax+by,\ cx+dy)$

& L sends lines to lines.

This all goes through a more general setting.

<u>Definition</u>: A mapping (or function)

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ is } \underline{\text{linear}} \text{ if}$$
$$: v \longmapsto L(v)$$

$$\text{(i) } L(u+v) = L(u) + L(v) \quad \forall\, u,v \in \mathbb{R}^n$$

$$\text{\& (ii) } L(rv) = rL(v), \quad \forall\, v \in \mathbb{R}^n,\ r \in \mathbb{R}.$$

Again, using (i) & (ii) we see that if

$$v = (x_1, x_2, \dots, x_n) = x_1e_1 + x_2e_2 + \dots + x_ne_n$$

where $e_i = (0,0,\dots,0,\underset{\substack{\uparrow \\ i\text{th position.}}}{1},0,\dots,0)$,

Then $$L(v) = x_1L(e_1) + \dots + x_nL(e_n)$$
$$= \sum_{i=1}^{n} x_iL(e_i).$$

So L is completely determined by the vectors $L(e_1), \dots, L(e_n) \in \mathbb{R}^m$.

As each $L(e_i)$ is a vector in $\mathbb{R}^m$, (n of them in all), i.e., m real numbers, we will label them as follows:

1 $L(e_1) := (a_{11}, a_{21}, a_{31}, \dots, a_{m1}) \in \mathbb{R}^m$.

2 $L(e_2) := (a_{12}, a_{22}, a_{32}, \dots, a_{m2}) \in \mathbb{R}^m$.

$\vdots$

n $L(e_n) := (a_{1n}, a_{2n}, a_{3n}, \dots, a_{mn}) \in \mathbb{R}^m$.

and store this defining information of L as an $m \times n$ matrix where $L(e_i)$ is the i^{th} column.

$$\begin{array}{c}
\begin{array}{cccc} 1 & 2 & & n \end{array} \\
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\end{array}$$

Remark: When we write, for instance,

$$L(e_1) = (a_{11}, a_{21}, \ldots, a_{m1}) \in \mathbb{R}^m$$
$$\| $$
$$a_{11}e_1 + a_{21}e_2 + \cdots + a_{m1}e_m$$

where here,
$$\left.\begin{aligned} e_1 &= (1,0,0,\ldots,0) \\ e_2 &= (0,1,0,\ldots,0) \\ &\vdots \\ e_m &= (0,0,\ldots,0,1) \end{aligned}\right\} \in \mathbb{R}^m \quad (m \text{ entries})$$

then implicitly the matrix we get involved using the basis or axes

$$\underbrace{(1,0,\ldots,0)}_{n \text{ entries}}, \ldots, (0,0,\ldots,0,1) \in \mathbb{R}^n$$

for $\mathbb{R}^n$, & the basis or axes

$$\underbrace{(1,0,\ldots,0)}_{m \text{ entries}}, \ldots, (0,\ldots,0,1) \in \mathbb{R}^m$$

for $\mathbb{R}^m$. For this reason we sometimes say that

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

is the matrix for L with respect to the <u>standard Basis</u> (as above) for $\mathbb{R}^n$ and $\mathbb{R}^m$.

Observe:

(i) A linear transformation $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ sends lines to lines because given a line ℓ in $\mathbb{R}^n$ (in parametric form)

$$\ell : \{ P + tv, \mid t \in \mathbb{R} \}$$

Then

$$L(\ell) = \{ L(P) + L(tv) \mid t \in \mathbb{R} \}$$
$$= \{ L(P) + t\,L(v) \mid t \in \mathbb{R} \}.$$

This is the parametric form of the line (in $\mathbb{R}^m$) through the point $L(P)$ in the direction $L(v)$.

(if $L(v) \neq 0$ $(= (0,0,0,\ldots,0))$)

(ii) L maps the zero vector in $\mathbb{R}^n$ to the zero vector in $\mathbb{R}^m$.

Because

$$\begin{array}{ccc} L(0+0) & = & L(0) \\ \| & & \\ L(0)+L(0) & = & L(0) \end{array}$$

$$\Longrightarrow L(0) = 0$$

Alternatively, using the matrix A associated to L.

$L(0)$ is obtained as $A \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^m$

Examples of Linear Transformations.

Ex: Fix a vector $n = (n_1, n_2, n_3) \in \mathbb{R}^3$

Define $L: \mathbb{R}^3 \to \mathbb{R}^3$ as follows:

$$L: v \longmapsto n \times v$$

"cross product"

L is linear, as $n \times (v+w) = (n \times v) + (n \times w)$

& $n \times (kv) = k(n \times v)$, $k \in \mathbb{R}$.

To find the matrix A_L for L (with respect to the standard basis $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ & $e_3 = (0,0,1)$) we find $L(e_1)$, $L(e_2)$ & $L(e_3)$ and then

$$A_L = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ L(e_1) & L(e_2) & L(e_3) \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

$$L(e_1) = n \times e_1 = \begin{vmatrix} e_1 & e_2 & e_3 \\ n_1 & n_2 & n_3 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= e_1\left((n_2)(0) - (n_3)(0)\right) - e_2\left((n_1)(0) - (n_3)(1)\right)$$
$$+ e_3\left((n_1)(0) - (n_2)(1)\right)$$

$$= n_3 e_2 - n_2 e_3 = (0, n_3, -n_2)$$

So the first column of A_L is $\begin{pmatrix} 0 & & \\ n_3 & & \\ -n_2 & & \end{pmatrix}$.

Next, $L(e_2) = n \times e_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ n_1 & n_2 & n_3 \\ 0 & 1 & 0 \end{vmatrix}$

$$= e_1\big((n_2)(0) - (n_3)(1)\big) - e_2\big((n_1)(0) - (n_3)(0)\big) + e_3\big((n_1)(1) - (n_2)(0)\big)$$

$$= -n_3 e_1 + n_1 e_3 = (-n_3, 0, n_1)$$

So the second column of A_L is $\begin{pmatrix} 0 & -n_3 & \\ n_3 & 0 & \\ -n_2 & n_1 & \end{pmatrix}$

Finally, $L(e_3) = n \times e_3 = \begin{vmatrix} e_1 & e_2 & e_3 \\ n_1 & n_2 & n_3 \\ 0 & 0 & 1 \end{vmatrix}$

$$= e_1\big((n_2)(1) - (n_3)(0)\big) - e_2\big((n_1)(1) - (n_3)(0)\big) + e_3\big((n_1)(0) - (n_2)(0)\big)$$

$$= n_2 e_1 - n_1 e_2 = (n_2, -n_1, 0)$$

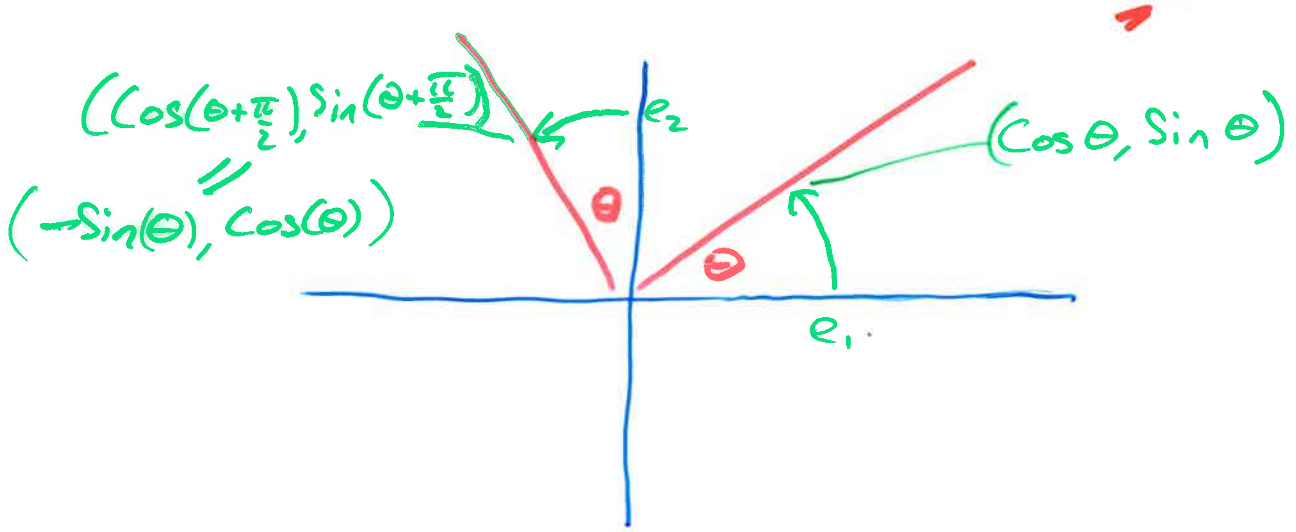
So the third column of A_L is $\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$

A_L is an example of a skew-symmetric matrix, i.e., $A_L^T = -A_L$ Transpose of A_L.

So $$A_L^T = \begin{pmatrix} 0 & n_3 & -n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} = -A_L$$

Ex: Let $L: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be rotation about the e_3 axis, "anticlockwise" by an angle θ. Then in the e_1-e_2 plane we see that $L(e_1) = \cos(\theta)e_1 + \sin(\theta)e_2 + 0e_3$.
& $L(e_2) = -\sin(\theta)e_1 + \cos(\theta)e_2$

"2D-picture"



Finally, $L(e_3) = e_3$, since e_3 is fixed by L.

$$\Rightarrow A_L = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$