Another application of solving a simple system of 2 linear equations gives the definition of the cross product of 2 vectors. $u \times v$ where $u = (u_1, u_2, u_3)$ & $v = (v_1, v_2, v_3) \in \mathbb{R}^3$.

Recall that the cross product of u & v is the vector denoted by $u \times v$ & it is perpendicular to both u & v.

i.e., $u \cdot (u \times v) = 0 = (u \times v) \cdot v$.

So if $u \times v = (x_1, x_2, x_3)$ then

(i) $u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$ &

(ii) $v_1 x_1 + v_2 x_2 + v_3 x_3 = 0$

$$\Rightarrow \left(\begin{array}{ccc|c} u_1 & u_2 & u_3 & 0 \\ v_1 & v_2 & v_3 & 0 \end{array}\right)$$

$$\xrightarrow[R_2 \mapsto v_2 R_1 - u_2 R_2]{R_1 \mapsto v_1 R_1 - u_1 R_2} \begin{pmatrix} 0 & \overbrace{v_1 u_2 - u_1 v_2}^{a} & \overbrace{v_1 u_3 - u_1 v_3}^{b} \\ \underbrace{u_1 v_2 - v_1 u_2}_{-a} & 0 & \underbrace{v_2 u_3 - u_2 v_3}_{c} \end{pmatrix}$$

$R_1: \; a x_2 = -b x_3$ (x_3 free).

$R_2: \; -a x_1 = -c x_3$

So $x_2 = -\frac{b}{a} x_3$, $x_1 = \frac{c}{a} x_3$.

Choose $x_3 = -a$ to get soln $x_1 = -c, x_2 = b$.

i.e., $(x_1, x_2, x_3) = (u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - v_1 u_2)$

$= u \times v$.

Mnemonic: Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$.

The cross product $u \times v$ of $u = (u_1, u_2, u_3)$ & $v = (v_1, v_2, v_3) \in \mathbb{R}^3$, can be computed by evaluating the following determinant

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{array}{l} e_1(u_2v_3 - v_2u_3) \\ -e_2(u_1v_3 - v_1u_3) \\ +e_3(u_1v_2 - v_1u_2) \end{array}$$

$$= ((u_2v_3 - v_2u_3), -(u_1v_3 - v_1u_3), (u_1v_2 - v_1u_2))$$

$$= u \times v.$$

E.g. $u = (2,1,0)$, $v = (3,2,1)$

$$u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = \begin{array}{l} e_1(1-0) \\ -e_2(2-0) \\ +e_3(4-3) \end{array}$$

$$= e_1 - 2e_2 + e_3 = (1,-2,1)$$

Note $(2,1,0) \cdot (1,-2,1) = 0$

& $(3,2,1) \cdot (1,-2,1) = 0$

Matrices and Linear Transformations

Recall: A <u>linear transformation</u> of the plane $\mathbb{R}^2$ is defined as a map or function $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that
$$v \longmapsto L(v)$$

(i) $L(u+v) = L(u) + L(v), \quad \forall\, u, v \in \mathbb{R}^2$

(ii) $L(rv) = rL(v), \quad \forall\, v \in \mathbb{R}^2, \ \& \ r \in \mathbb{R}.$

Therefore if $v = (x, y) = x(1,0) + y(0,1)$.

Then $L(v) = L(x(1,0) + y(0,1))$
$$= xL(1,0) + yL(0,1)$$

So L is determined by $L((1,0)) = (a, c)$
& $L((0,1)) = (b, d)$

where $a, b, c, d \in \mathbb{R}$.

We encode these 4 numbers in a 2×2 matrix, i.e, an array of 2 rows and 2 columns
$$\begin{pmatrix} \uparrow & \uparrow \\ L(1,0) & L(0,1) \\ \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Thus $L(v) = L((x,y))$ is obtained as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

i.e., $L(v) = (ax+by,\ cx+dy)$

& L sends lines to lines.

This all goes through a more general setting.

<u>Definition</u>: A mapping (or function)

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ is } \underline{\text{linear}} \text{ if}$$
$$: v \longmapsto L(v)$$

(i) $L(u+v) = L(u) + L(v) \quad \forall\, u, v \in \mathbb{R}^n$.

& (ii) $L(rv) = rL(v), \quad \forall\, v \in \mathbb{R}^n,\ r \in \mathbb{R}$.

Again, using (i) & (ii) we see that if

$v = (x_1, x_2, \ldots, x_n) = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$

where $e_i = (0,0,\ldots,0,\underset{\substack{\uparrow \\ i\text{th position.}}}{1},0,\ldots,0)$,

Then $L(v) = x_1 L(e_1) + \cdots + x_n L(e_n)$

$$= \sum_{i=1}^{n} x_i L(e_i).$$