Section 1.4.3 : Partial Fraction Expansions

We know how to integrate polynomial functions; for example

$$\int 2x^2+3x-4\,dx = \frac{2}{3}x^3+\frac{3}{2}x^2-4x+C.$$

We also know that

$$\int \frac{1}{x}\,dx = \ln|x|+C$$

and that

$$\int \frac{1}{x^n}\,dx = -\frac{1}{n-1}\frac{1}{x^{n-1}}+C,$$

for $n>1$.
This section is about integrating rational functions; i.e. quotients in which the numerator and denominator are both polynomials.

Adding Symbolic Fractions

Remark: If we were presented with the task of adding the expressions $\frac{2}{x+3}$ and $\frac{1}{x+4}$, we would take $(x+3)(x+4)$ as a common denominator and write

$$\frac{2}{x+3}+\frac{1}{x+4} = \frac{2(x+4)}{(x+3)(x+4)}+\frac{1(x+3)}{(x+3)(x+4)} = \frac{2(x+4)+1(x+3)}{(x+3)(x+4)} = \frac{3x+11}{(x+3)(x+4)}.$$

Question: Suppose we were presented with the expression $\frac{3x+11}{(x+3)(x+4)}$ and asked to rewrite it in the form $\frac{A}{x+3}+\frac{B}{x+4}$, for *numbers* A and B. How would we do it?

Another Question Why would we want to do such a thing?

The Partial Fraction Expansion

Write

$$\frac{3x+11}{(x+3)(x+4)} = \frac{A}{x+3}+\frac{B}{x+4}.$$

Then

$$\frac{3x+11}{(x+3)(x+4)} = \frac{A(x+4)}{(x+3)(x+4)}+\frac{B(x+3)}{(x+3)(x+4)} = \frac{(A+B)x+4A+3B}{(x+3)(x+4)}.$$

This means $3x+11=(A+B)x+4A+3B$ for all x, which means

$$A+B=3, \text{ and } 4A+3B=11.$$

Thus $B=1$ and $A=2$. So

$$\frac{3x+11}{(x+3)(x+4)} = \frac{2}{x+3}+\frac{1}{x+4}.$$

An Alternative Method

We want

$$3x+11 = A(x+4)+B(x+3),$$

for all real numbers x. If this statement is true for all x, then in particular it is true when $x=-4$. Setting $x=-4$ gives

$$-12+11 = A(0)+B(-1) \implies B=1.$$

Setting $x=-3$ gives

$$-9+11 = A(1)+B(0) \implies A=2.$$

Thus

$$\frac{3x+11}{(x+3)(x+4)} = \frac{2}{x+3}+\frac{1}{x+4}.$$

Integration using partial fractions

Example 28

Determine $\int \frac{3x+11}{(x+3)(x+4)}\,dx$.

Solution : Write

$$\int \frac{3x+11}{(x+3)(x+4)}\,dx = \int \frac{2}{x+3}\,dx + \int \frac{1}{x+4}\,dx$$

Then

$$\int \frac{3x+11}{(x+3)(x+4)}\,dx = 2\ln|x+3|+\ln|x+4|+C = \ln((x+3)^2)+\ln|x+4|+C.$$

Partial fractions with long division

Example 29

Determine $\int \frac{x^3+3x+2}{x+1}\,dx$.

In this example the degree of the numerator exceeds the degree of the denominator, so first apply long division to find the quotient and remainder upon dividing x^3+3x+2 by $x+1$.
We find that the quotient is x^2-x+4 and the remainder is -2. Hence

$$\frac{x^3+3x+2}{x+1} = x^2-x+4+\frac{-2}{x+1}.$$

Thus

$$\int \frac{x^3+3x+2}{x+1}\,dx = \int x^2-x+4\,dx - 2\int \frac{1}{x+1}\,dx = \frac{1}{3}x^3-\frac{1}{2}x^2+4x-2\ln|x+1|+C.$$

A Harder Example

Example 30

Determine $\displaystyle\int \frac{x+1}{(2x+1)^2(x-2)}\,dx.$

Solution: In this case the denominator has a repeated linear factor $2x+1$. It is necessary to include both $\dfrac{A}{2x+1}$ and $\dfrac{B}{(2x+1)^2}$ in the partial fraction expansion. We have

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{x-2}.$$

Then

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{A(2x+1)(x-2)+B(x-2)+C(2x+1)^2}{(2x+1)^2(x-2)}.$$

and so

$$x+1 = A(2x+1)(x-2)+B(x-2)+C(2x+1)^2.$$

A Harder Example

$$x=2: \quad 3 = C(5)^2 \qquad C = \tfrac{3}{25}$$

$$x=-\tfrac{1}{2}: \quad \tfrac{1}{2} = B\left(-\tfrac{5}{2}\right) \qquad B = -\tfrac{1}{5}$$

$$x=0: \quad 1 = A(1)(-2)+B(-2)+C(1)^2 \quad A = -\tfrac{6}{25}$$

Thus

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{-6/25}{2x+1} + \frac{-1/5}{(2x+1)^2} + \frac{3/25}{x-2}$$

and

$$\int \frac{x+1}{(2x+1)^2(x-2)}\,dx = -\frac{6}{25}\int \frac{1}{2x+1}\,dx - \frac{1}{5}\int \frac{1}{(2x+1)^2}\,dx + \frac{3}{25}\int \frac{1}{x-2}\,dx.$$

A Harder Example

Call the three integrals on the right above I_1, I_2, I_3 respectively.

- I_1 : $\displaystyle\int \frac{1}{2x+1}\,dx = \frac{1}{2}\ln|2x+1|(+C_1).$
- I_2 : $\displaystyle\int \frac{1}{(2x+1)^2}\,dx = -\frac{1}{2(2x+1)}(+C_2).$
- I_3 : $\displaystyle\int \frac{1}{x-2}\,dx = \ln|x-2|(+C_3).$

Thus

$$\int \frac{x+1}{(2x+1)^2(x-2)}\,dx = -\frac{3}{25}\ln|2x+1| + \frac{1}{10(2x+1)} + \frac{3}{25}\ln|x-2| + C.$$

Learning outcomes for Section 1.4

At the end of this section you should

- Know the difference between a definite and indefinite integral and be able to explain it accurately and precisely.
- Be able to evaluate a range of definite and indefinite integrals using the following methods:
  - direct methods;
  - suitably chosen substitutions;
  - integration by parts;
  - partial fraction expansions.