Lecture 8 Complex Numbers

Suppose the only numbers we had were

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\} \quad \text{(the positive integers)}.$$

With $\mathbb{N}$, we can easily solve equations like $x - 5 = 0$, $x - 17 = 0$, $x - n = 0$ where $n \in \mathbb{N}$. Their solutions all exist in $\mathbb{N}$ (eg $5, 17, n$).

But if we want to solve $x + 5 = 0$, say, then we're stuck if we only have $\mathbb{N}$ at our disposal.

We can however overcome our problem by widening our set of available numbers to $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$ (all integers).

Now we can readily solve equations of the form $x - 5 = 0$ and $x + 5 = 0$ etc with $\mathbb{Z}$ available to us ..... great!

But if we have to solve an equation like $2x - 5 = 0$, we've back to square one ...... unless we enlarge $\mathbb{Z}$ to a set that includes numbers like $5/2$ ie fractions.

So now we need $\mathbb{Q} = \{m/n : m \in \mathbb{Z}, n \in \mathbb{N}\}$ (the rational numbers).

And so the number systems evolve from $\mathbb{N}$ to $\mathbb{Z}$ to $\mathbb{Q}$ as the (simple) equations we want to solve demand.

The simple equation $x^2 = 2$ however has no solution in $\mathbb{Q}$ because the numbers that solve it, namely $\sqrt{2}$ and $-\sqrt{2}$ are <u>not</u> in $\mathbb{Q}$. They are irrational - so we next have to add all irrational numbers into the mix ..... and this gives us $\mathbb{R}$, the set of all real numbers.

And yet problems still persist in solving simple equations. If x is a real number, then x^2 is never negative ie $x^2 \geqslant 0$.
So once again our number system $\mathbb{R}$ won't let us solve $x^2+1=0$.

Introducing ..... i. i satisfies $i^2+1=0$
ie $i^2=-1$ so clearly $i \notin \mathbb{R}$.
We define a complex number z to be an expression of the form $z=x+iy$ where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.
We also define arithmetic for complex numbers:
addition/subtraction: $(x+iy)\pm(a+ib) = (x\pm a)+i(y\pm b)$
multiplication: $(x+iy)(a+ib) = (xa-yb)+i(ya+xb)$

$$= x(a+ib)+iy(a+ib)$$
$$= xa+ibx+iya+\underbrace{i^2}_{=-1}yb = (xa-yb)+i(ya+bx)$$

The arithmetic is based on treating $x+iy$ as a 'normal' algebraic expression, using $i^2=-1$ wherever it arises.
So multiplication of $x+iy$ by $a+ib$ inevitably is $(xa-yb)+i(ya+xb)$ just by multiplying out the brackets as we saw above.

Examples
If $z_1=2+3i$ and $z_2=3-5i$, then z_1+z_2
$$= (2+3i)+(3-5i)$$
$$= (2+3)+i(3-5) = 5-2i$$
We call 5 the real part of $5-2i$;
we call -2 " imaginary part of $5-2i$.
More generally, if $z=x+iy$, then the real part of z is x, while the imaginary part of z is y.

Let's calculate $z_1 - z_2$: $z_1 = 2+3i$, $z_2 = 3-5i$

$$\Rightarrow z_1 - z_2 = (2+3i) - (3-5i)$$
$$= 2 + 3i - 3 + 5i$$
$$= -1 + 8i.$$

$$z_1 z_2 = (2+3i)(3-5i) = 2(3-5i) + 3i(3-5i)$$
$$= 6 - 10i + 9i - 15i^2 = 6 - i - 15(-1)$$
$$= 21 - i.$$

So that's addition, subtraction and multiplication sorted... but what about division? How do we calculate $\frac{z_1}{z_2}$?

Let's look at a few examples:

(i) $\dfrac{8+12i}{4} = \dfrac{8}{4} + \dfrac{12i}{4} = 2+3i$

(ii) $\dfrac{8+12i}{3.7} = \dfrac{8}{3.7} + \dfrac{12}{3.7}i$

Dividing by a real number is not a problem.

But how do we make sense of $\frac{2+3i}{3-5i}$ as a complex number? How can we write it as an expression of the form $x+iy$?

Noting that $(a+ib)(a-ib) = a^2 - (ib)^2 = a^2 - (i^2)b^2$
(difference of 2 squares) $= a^2 + b^2$,

we can rewrite $\dfrac{2+3i}{3-5i}$ as $\dfrac{2+3i}{3-5i} \cdot \underbrace{\left(\dfrac{3+5i}{3+5i}\right)}_{1}$ $= \dfrac{(2+3i)(3+5i)}{34}$

" (so no change)

so now division (by a real number) is straightforward

$$= \frac{6+10i+9i+15i^2}{34} = \frac{6+19i-15}{34} = \frac{-9+19i}{34} = \frac{-9}{34} + \frac{19}{34}i$$

So $\dfrac{2+3i}{3-5i} = -\dfrac{9}{34} + \dfrac{19}{34}i$.

We denote by $\mathbb{C}$ the set of all complex numbers

ie $\mathbb{C} = \{x+iy : x, y \in \mathbb{R}\}$.