Lecture 18

Example: Solve the system:

$$\left.\begin{array}{l} 1\cdot w + 3x + 3y + 2z = 1 \\ 2w + 6x + 9y + 5z = 5 \\ -w - 3x + 3y = 5 \end{array}\right\} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Solution: Keep R_1 fixed, and use 1 as pivot:

$$\begin{array}{rl} & w + 3x + 3y + 2z = 1 \\ R_2 - 2R_1 \to & 0 + 0 + 3y + z = 3 \\ R_3 + R_1 \to & 0 + 0 + 6y + 2z = 6 \end{array}$$

Now use 3 as pivot:

Keep R_1 and R_2 fixed and use 3 in R_2 as pivot:

$$\left.\begin{array}{rl} & w + 3x + 3y + 2z = 1 \\ & 3y + z = 3 \\ R_3 - 2R_2 \to & 0y + 0z = 0 \end{array}\right\} \leftarrow$$

indicates that 3^{rd} equation is 'redundant'.

The original system has the same solutions as the simplified system:

$$\left.\begin{array}{l} w + 3x + 3y + 2z = 1 \\ 3y + z = 3 \end{array}\right\}$$

dependent variables

Let's take $z = t$, and take $x = s$.

Then $3y = 3 - t$ so $y = 1 - \frac{1}{3}t$

and $w = 1 - 3s - 3(1 - \frac{1}{3}t) - 2t$

$= -2 - 3s - t$

The solutions are:

$$\text{dependent variables} \to \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2-3s-t \\ s \\ 1-\frac{1}{3}t \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{pmatrix}$$

These solutions form "a plane" in $\mathbb{R}^4$. More precisely, we have a 2-dimensional (because of the parameters s and t) solution space.

Terminology: The first nonzero entries in the rows of the simplified system are called dependent variables. So above, w and y are dependent. The remaining variables are then called independent. So above, x and z are independent variables.

Problem: Solve

$$\left.\begin{aligned} w - 2x + y - z &= 4 \\ 2w - 3x + 2y - 3z &= -1 \\ 3w - 5x + 3y - 4z &= 3 \\ -w + x - y + 2z &= 5 \end{aligned}\right\}$$

$$\begin{aligned} & & w - 2x + y - z &= 4 & &① \\ R_2 - 2R_1 &\to & 0 + 1x + 0y - z &= -9 & &② \quad \text{Use pivot } 1. \\ R_3 - 3R_1 &\to & 0 + 1\cdot x + 0y - z &= -9 \\ R_4 + R_1 &\to & 0 - x + 0y + z &= 9 \end{aligned}$$

$$\left.\begin{aligned} w - 2x + y - z &= 4 \\ x - z &= -9 \end{aligned}\right\}$$

$$\left.\begin{aligned} R_3 - R_2 &\to & 0 + 0 &= 0 \\ R_4 + R_2 &\to & 0 + 0 &= 0 \end{aligned}\right\} \text{ redundant equations}$$

Effectively we have a system of 2 equations in 4 unknowns (w, x, y, z).
Dependent variables are w, x.
Independent (or free) variables are y, z.

Let $z = t$, $y = s$:

Equation ② gives $x - t = -9 \Rightarrow x = t - 9$

Equation ① gives $w - 2(t-9) + s - t = 4$

ie $w - 2t + 18 + s - t = 4$

ie $w = -14 - s + 3t$

$$\therefore \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -14 - s + 3t \\ t - 9 \\ s \\ t \end{pmatrix} = \begin{pmatrix} -14 \\ -9 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

for all $s, t \in \mathbb{R}$.

(We have a 2-dimensional solution space.)

Comment: Different row operations may result in different independent variables – but it is a theorem (ie a fact) that the number of independent variables is the same regardless.

Problem Find all values of K for which the following system has no solutions:

$$\left.\begin{aligned} x + Ky &= 0 \\ Kx + 9y &= 1 \end{aligned}\right\}$$ Two lines in $\mathbb{R}^2$

Solution:

$$\left.\begin{aligned} & & x + Ky &= 0 \\ R_2 - KR_1 &\rightarrow & 0x + (9-K^2)y &= 1 \end{aligned}\right\}$$

The (new) second equation $(9-K^2)y = 1$ shows that if $9 - K^2 = 0$, then $0 = 1$ so that there are no solutions (since $0.y = 1$ cannot be solved for any $y \in \mathbb{R}$)

ie if $K^2 = 9$, there are no solutions

ie if $K = 3$ or $K = -3$, there are no solutions.

(Note that if $K^2 \neq 9$, then $9 - K^2 \neq 0$ and so $y = \frac{1}{9-K^2}$

$$x = -Ky = \frac{-K}{9-K^2} = \frac{K}{K^2-9}$$

are the solutions.)

Look at what happens when $K = -3$: our system is

$$x - 3y = 0 \quad \Rightarrow \quad y = \tfrac{1}{3}x$$

$$-3x + 9y = 1 \quad \Rightarrow \quad y = \tfrac{1}{9}(1 + 3x) = \tfrac{1}{3}x + \tfrac{1}{9}$$

So the lines are parallel (and do not cross – so no solution.)