Lecture 16

Terminology: 2 is the pivot at the first stage; $-\frac{1}{2}$ is the pivot at the second stage.

Example: Solve the linear system:

$$\left.\begin{array}{l} \overset{\text{pivot}}{x} + y + z = -2 \\ 3x + y - z = 6 \\ x - y + z = -1 \end{array}\right\} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\left.\begin{array}{lr} R_1 & x + y + z = -2 \\ R_2 \to R_2 - 3R_1 : & 0 + 0 - 4z = 12 \\ R_3 \to R_3 - R_1 : & 0 - 2y + 0 = 1 \end{array}\right\}$$

We have

$$\left.\begin{array}{r} x + y + z = -2 \\ -4z = 12 \\ -2y \quad = 1 \end{array}\right\} \text{ ie } \xrightarrow{R_2 \leftrightarrow R_3} \left.\begin{array}{r} x + y + z = -2 \\ -2y \quad = 1 \\ -4z = 12 \end{array}\right\}$$

swap row 2 (R_2) with row 3 (R_3)

This simpler system is said to be in row echelon form

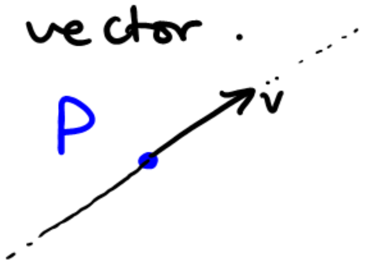
Solve now! $-4z = 12 \Rightarrow z = -3$

$$-2y = 1 \Rightarrow y = -\tfrac{1}{2}$$

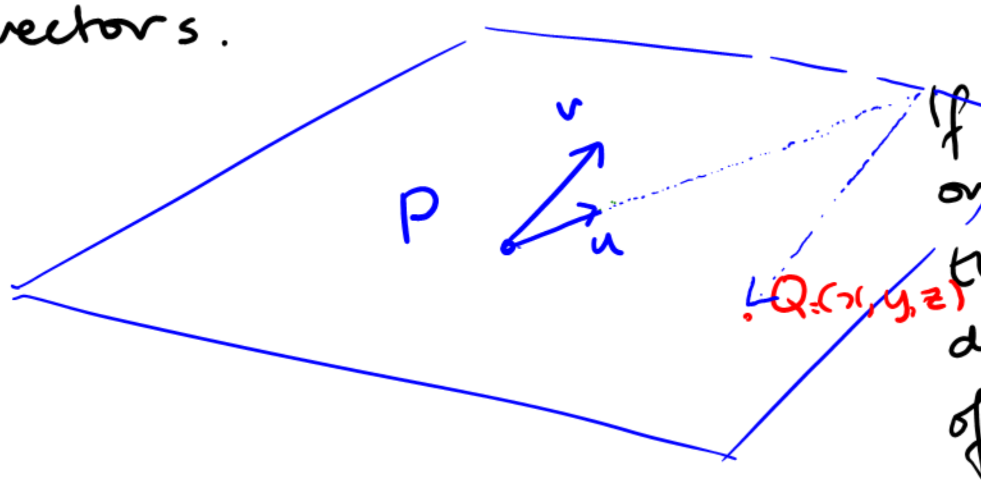
$$x + (-\tfrac{1}{2}) + (-3) = -2$$

$$\Rightarrow x = 1\tfrac{1}{2}.$$

Some geometry: a line in $\mathbb{R}^3$ is determined by a point P on the line and a direction vector.



A plane in $\mathbb{R}^3$ is determined by a point P on the plane and 2 direction vectors.



If Q is any point on the plane, then it can be described in terms of P, v and u.

In $\mathbb{R}^3$, $su + tv$ (where $s \in \mathbb{R}, t \in \mathbb{R}$) spans a plane through the origin $(0,0,0)$ while $P + su + tv$ spans a plane through P.

eg $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix}}_{P} + s\underbrace{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}_{u} + t\underbrace{\begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}}_{v}$$ is a plane through $(0,0,7)$

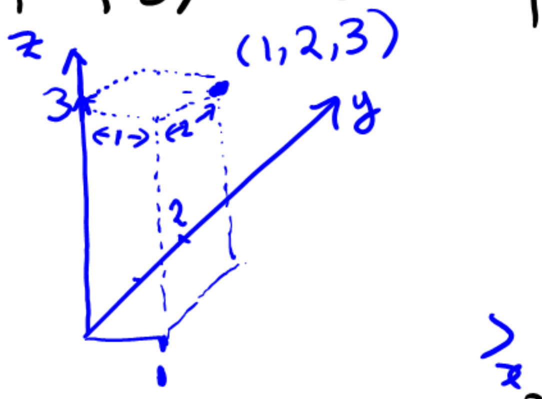
In general, a linear equation
$$ax + by + cz = d$$
describes a plane in $\mathbb{R}^3$.

The link between the 'vector equation of a plane' $P + su + tv$ (where P is a point on the plane and u and v are 2 direction vectors in the plane) and the algebraic form
$$ax + by + cz = d$$
comes from setting $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ equal to $P + su + tv$, and solving.

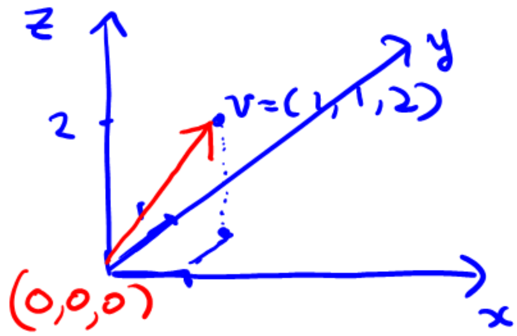
Brief review: In $\mathbb{R}^3$, a point P is a triple

ie $P = (x, y, z)$ for some real numbers x, y, z.

Example: $(1, 2, 3)$ is a point in $\mathbb{R}^3$.



A **direction vector** v in $\mathbb{R}^3$ is also a triple, for example, $v = (1, 1, 2)$



Example Let $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 3x + 5y + 7z = 15 \right\} \subseteq \mathbb{R}^3$.

We shall show that S, ie the solutions $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ that satisfy $3x + 5y + 7z = 15$, is a plane by showing it has the form $P + su + tv$ for some point P in S and direction vectors u, v.

First, find a point P in S - any point will do. For example, $P = \begin{pmatrix} 15/3 \\ 0 \\ 0 \end{pmatrix}$ (by letting $y = 0 = z$) lies in S (as do $\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 15/7 \end{pmatrix}$ etc.)

If we let $y = s$ and $z = t$, then

$$3x = 15 - 5y - 7z \Rightarrow x = 5 - \tfrac{5}{3}y - \tfrac{7}{3}z$$

$$\text{ie } x = 5 - \tfrac{5}{3}s - \tfrac{7}{3}t .$$

Thus an arbitrary point $Q = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in S has the form

$$Q = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 - \frac{5}{3}s - \frac{7}{3}t \\ s \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}}_{\text{point } P} + s\begin{pmatrix} -5/3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -7/3 \\ 0 \\ 1 \end{pmatrix}$$

direction vectors

ie Q takes the form $P + su + tv$ where u and v are the given direction vectors

Thus, S is indeed a plane.

CONCLUSION

In general, a linear equation

$$ax + by + cz = d$$

determines a plane in $\mathbb{R}^3$.