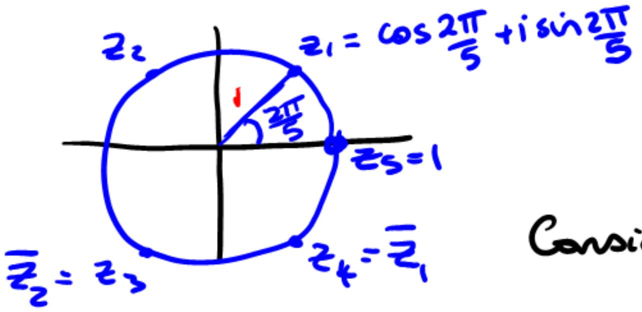
Lecture 14

Notice also that $z_1^2 = z_2$, $z_1^3 = z_3$, $z_1^4 = z_4$
(and of course $z_1^5 = 1$!)

Back to the factorisation of $x^5 - 1$:
we have

$$x^5 - 1 = (x-1)\underbrace{(x-z_2)(x-z_3)(x-z_1)(x-z_4)}$$

all have non-zero imaginary part
so these are <u>not</u> <u>real</u> factors



Consider $(x-z_2)(x-z_3)$

$$= x^2 - (z_2 + z_3)x + z_2 z_3$$

$$= x^2 - \underbrace{(z_2 + \overline{z}_2)}_{\text{a real number}} x + \underbrace{z_2\overline{z}_2}_{\text{a real number}}$$

(see note at start of lecture 13)

Recall: $z_3 = \overline{z}_2$.

$$\begin{cases} z_2 + \overline{z}_2 = 2\cos\frac{4\pi}{5} \\ z_2 . \overline{z}_2 = |z_2|^2 = 1 \end{cases}$$

The same happens for $(x-z_1)(x-z_4)$

ie $(x-z_1)(x-z_4) = (x-z_1)(x-\overline{z}_1)$

Recall: $z_4 = \overline{z}_1$

$$= x^2 - (z_1 + \overline{z}_1)x + z_1\overline{z}_1$$

$$= x^2 - \left(2\cos\frac{2\pi}{5}\right)x + |z_1|^2$$

$$= x^2 - \left(2\cos\frac{2\pi}{5}\right)x + 1$$

Putting it all together,
we have $x^5 - 1 = (x-1)\left(x^2 - 2\cos\frac{2\pi}{5}x + 1\right)\left(x^2 - 2\cos\frac{4\pi}{5} + 1\right)$
is the factorisation in terms of real linear/quadratic factors.

Recall that $z = x + iy$ can be rewritten
as $z = r(\cos\theta + i\sin\theta)$
where $r = |z|$, $\theta = \arg(z)$.

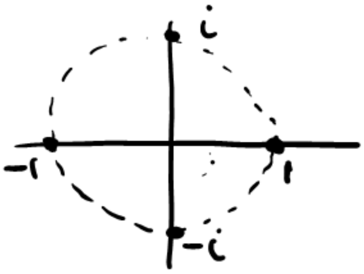
"Notation"
Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

$e^0 = 1$

$e^{i\pi} = \cos\pi + i\sin\pi = -1 + i0 = -1$

$e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = 0 + i1 = i$

$e^{-i\pi/2} = -i$ (rotation $\pi/2$ clockwise)



Recall in $\mathbb{R}$: $\begin{cases} a^x \cdot a^y = a^{x+y} \\ \dfrac{a^x}{a^y} = a^{x-y} \\ a^0 = 1 \end{cases}$

Further, we learnt that

phi

$(\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi)$

theta $= \cos(\theta+\varphi) + i\sin(\theta+\varphi)$

ie $e^{i\theta} e^{i\varphi} = e^{i(\theta+\varphi)}$ same index laws as for $\mathbb{R}$

Example: $\underset{\substack{\shortparallel \\ i}}{e^{i\pi/2}} \cdot \underset{\substack{\shortparallel \\ -1}}{e^{i\pi}} = e^{i\pi/2 + i\pi} = \underset{\substack{\shortparallel \\ -i}}{e^{i(\frac{3\pi}{2})}}$

Problem Find the two square roots of $2i$.

ie solve $z^2 = 2i$

$= 0 + 2i$

Solution

Here $|z^2| = 2$ so $|z|^2 = 2$ and so $|z| = \sqrt{2}$

$\operatorname{Arg}(z^2) = \pi/2 + 2\pi k$ for $k = 1, 2$

ie $2\arg(z) = \pi/2 + 2\pi k$ for $k = 1, 2$

$\operatorname{Arg}(z) = \pi/4 + \pi k$ for $k = 1, 2$.

See comment later about why it is enough to only take $k = 1, 2$.

So the two square roots of $2i$ are:

$z_1 = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \sqrt{2}e^{\frac{5\pi}{4}i}$ (here $k=1$)

and $z_2 = \sqrt{2}\left(\cos(\pi/4 + 2\pi) + i\sin(\pi/4 + 2\pi)\right) = \sqrt{2}(\cos\pi/4 + i\sin\pi/4) = \sqrt{2}e^{\frac{\pi}{4}i}$

(Note that z_1, z_2 is NOT a conjugate pair.) (here $k=2$)

Note the general approach for solving $z^n = w$ for any positive integer n and any complex number

1) write w in polar form ie in terms of its modulus r and its argument φ.

2) recognise that we can increase φ by adding 2π to it as many times as we like ie

$$\varphi + 2\pi K \qquad \text{for } K = 1, 2, \ldots$$

We will still have the same complex number w if we keep the same modulus (r) but take the argument as $\varphi + 2\pi K$ for any integer n.

3) Then z must have modulus $\sqrt[n]{r}$ (ie the n^{th} root of r) and its argument must be $\frac{1}{n}(\varphi + 2\pi K)$

(de Moivre's theorem tells us this) $= \varphi/n + \frac{2\pi K}{n}$ for $K = 1, \ldots, n$.

"Slice" the argument by n.

ie we will have exactly n solutions

$$z_1, z_2, z_3, \ldots, z_n,$$

one for each value of K.

Why? Read on....

4) The reason that we 'stop' K at n (ie only let K run from 1 up as far as n) is because the solution z_n at $n = K$

has an argument $\varphi/n + \frac{2\pi n}{n} = \frac{\varphi}{n} + 2\pi = \varphi/n$

If we then took $K = n+1$ to calculate another solution, its argument would be $\frac{\varphi}{n} + \frac{2\pi}{n}(n+1)$

$$= \varphi/n + 2\pi + \frac{2\pi}{n}$$

$$= \varphi/n + \frac{2\pi}{n}$$

$$= \varphi/n + \frac{2\pi(1)}{n}$$

which is the same argument as the solution z_1

So the solutions repeat after $K=n$, cycling back to z_1, z_2 etc.

Consider the above points in the context of the examples we have already worked through. In the case for $z^5=1$ (here, $w=1$), we saw that $r=1$ and we let $\theta = 0+2\pi K$ for $K=1,2,3,4,5$. ($n=5$)

Then by de Moivre's theorem, $|z|^5 = 1$

ie $|z| \left(= \sqrt[5]{1}\right) = 1$

while $\arg(z) = \dfrac{0+2\pi K}{5} = \dfrac{2\pi K}{5}$ for $K=1,2,3,4,5$.

So the five solutions/roots are:

$$z_K = \cos\frac{2\pi K}{5} + i\sin\frac{2\pi K}{5} \quad \text{for } K=1,2,3,4,5.$$

So z_1, z_2, z_3, z_4 and z_5 are our solutions.

If you compute z_K for $K=6,7,9,10,11,12,13,\dots$, you will find

$z_6 = z_1$, $z_7 = z_2$, $z_8 = z_3$, $z_9 = z_4$, $z_{10} = z_5$,

$z_{11} = z_1$, $z_{12} = z_2$, $z_{13} = z_3$, $z_{14} = z_4$, $z_{15} = z_5$,

etc.

The original five solutions $z_1, \dots, z_5$ repeat forever.

Note that $w=1$ is a very "easy" complex number of which to find roots because its modulus of 1 yields a modulus of 1 for the roots, while its argument of 0 or $0+2\pi K$ is similarly easy.

Thus we can readily solve

a) $z^6 = 1$ for z ; here we will "slice" or divide the argument $0+2\pi K$ by 6, so $z_K = \cos\frac{2\pi K}{6} + i\sin\frac{2\pi K}{6}$ for $K=1,\dots,6$

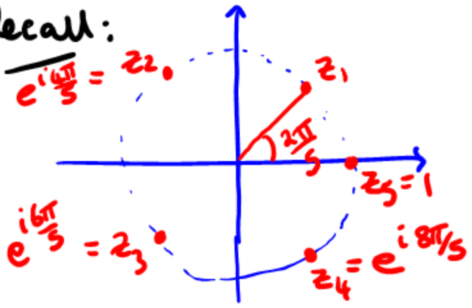
ie $z_K = e^{\frac{2\pi}{6}K}$ for $K=1,\dots,6$.

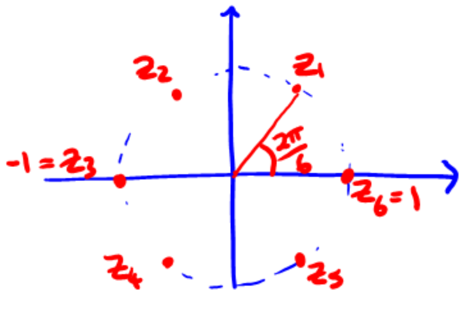
b) $z^7 = 1$; here we divide (again by de Moivre's Theorem) the argument $0+2\pi K$ by 7, so $z_K = \cos\frac{2\pi}{7}K + i\sin\frac{2\pi}{7}K$ for $K=1,\dots,7$.

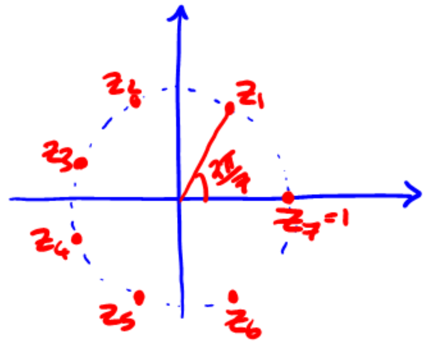
And so it continues ......

A quick check through the solutions for $z^6=1$ and for $z^7=1$ will confirm (as for $z^5=1$) that the non-real roots must occur in conjugate pairs.

Recall:







$z^5 = 1$	$z^6 = 1$	$z^7 = 1$
$\begin{cases} z_4 = \bar{z}_1 \\ z_3 = \bar{z}_2 \end{cases}$	$\begin{cases} z_5 = \bar{z}_1 \\ z_4 = \bar{z}_2 \end{cases}$	$\begin{cases} z_6 = \bar{z}_1 \\ z_5 = \bar{z}_2 \\ z_4 = \bar{z}_3 \end{cases}$

So $\sin\frac{8\pi}{5} = -\sin\frac{2\pi}{5}$

ie $\sin\frac{2\pi}{5} + \sin\frac{8\pi}{5} = 0$

Similarly, $\sin\frac{6\pi}{5} = -\sin\frac{4\pi}{5}$

ie $\sin\frac{4\pi}{5} + \sin\frac{6\pi}{5} = 0$

Thus,

$\sin\frac{2\pi}{5} + \sin\frac{4\pi}{5} + \sin\frac{6\pi}{5} + \sin\frac{8\pi}{5} = 0$.

The same idea works here:

$\sin\frac{10\pi}{6} + \sin\frac{2\pi}{6} = 0 \quad (\because z_5 = \bar{z}_1)$

$\sin\frac{8\pi}{6} + \sin\frac{4\pi}{6} = 0 \quad (\because z_4 = \bar{z}_2)$

It follows that

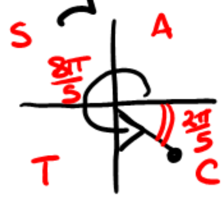
$\sin\frac{2\pi}{6} + \sin\frac{4\pi}{6} + \sin\frac{8\pi}{6} + \sin\frac{10\pi}{6} = 0$.

We could of course have worked this out directly using the fact that

$$\sin\frac{8\pi}{5} = -\sin\frac{2\pi}{5}$$

$$\sin\frac{6\pi}{5} = -\sin\pi/5$$

$$\sin\frac{4\pi}{5} = \sin\frac{\pi}{5}$$



$\sin\frac{8\pi}{5} = -\sin\frac{2\pi}{5}$

$$\therefore \sin\frac{2\pi}{5} + \sin\frac{8\pi}{5} = 0 = \sin\frac{4\pi}{5} + \sin\frac{6\pi}{5} \quad \text{as before.}$$

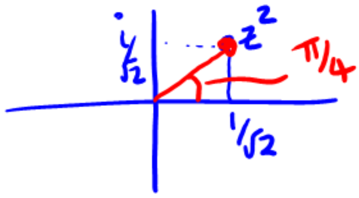
But it's nice to see it from another perspective, such as the five roots of unity, and the conjugate pairs amongst these. NOTE THAT THIS HAPPENS ONLY FOR ROOTS OF REAL NUMBERS.

Problem Determine the square roots of $\frac{1+i}{\sqrt{2}}$.

Solution. Here we want to solve $z^2 = \frac{1+i}{\sqrt{2}}$ for z.

(So $w = \frac{1}{\sqrt{2}}(1+i)$ this time.) Then $|z^2| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{1} = 1$

and $\text{Arg}(w) = \pi/4 + 2\pi k$ for $k = 1, 2$.



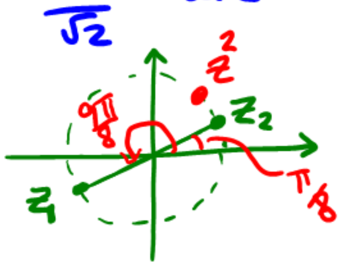
Further $\text{Arg}(z^2) = 2\,\text{Arg}(z) = \pi/4 + 2\pi k$ for $k = 1, 2$

and so $\text{Arg}(z) = \pi/8 + \pi k$ for $k = 1, 2$.

So the two square roots of $\frac{1+i}{\sqrt{2}}$ are

$z_1 = e^{i\frac{9\pi}{8}}$

$z_2 = e^{i\pi/8}$



So again z_1, z_2 is not a conjugate pair.