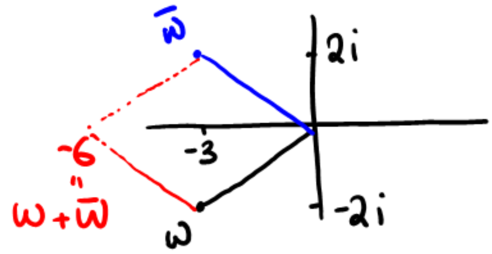
Lecture 13

Recall that the complex conjugate of $w = a+bi$ is $\bar{w} = a-bi$. So if $w = -3-2i$, then $w = -3+2i$.



Note that

$$w+\bar{w} = (a+bi)+(a-bi)$$
$$= a + a$$
$$= 2a \text{ ..... a } \underline{\text{real}} \text{ number.}$$

(In our example, $w+\bar{w} = 2(-3) = -6$.)

Note also that $w.\bar{w} = (a+bi)(a-bi) = a^2+b^2$
$(= |w|^2)$
..... also a real number.

So adding a complex number to its conjugate, and multiplying a complex number by its conjugate always yields a real number.

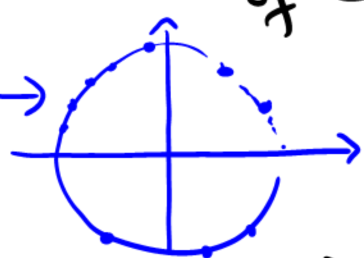
Keep this in mind as we go further.....

Problem Find all z such that $z^5 = 1$.
Then factorise the polynomial x^5-1 as
i) a product of linear factors
ii) a product of real linear/quadratic factors.

Solution: Since $z^5=1$, then $|z^5| = 1 = |z|^5$
$\Rightarrow |z|=1$

Thus z lies on the unit circle (ie the circle of centre $(0,0)$, radius 1).

But which of these points z satisfy $z^5 = 1$?

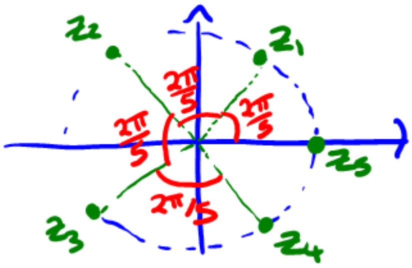


Again since $z^5=1$, $\operatorname{Arg}(z^5) = \operatorname{Arg}(1) = 0$ or 2π or 4π or .....

and by de Moivre's Theorem,
$$\operatorname{Arg}(z^5) = 5\operatorname{Arg}(z).$$

Thus $\quad 5\,\mathrm{Arg}(z) = 0$ or 2π or 4π or ...

$\therefore \quad \mathrm{Arg}(z) = 0$ or $\frac{2\pi}{5}$ or $\frac{4\pi}{5}$ or $\frac{6\pi}{5}$ or ....



Note: $2\pi = 360^\circ$

so $\pi = 180^\circ$

$\pi/2 = 90^\circ$

$\pi/5 = 36^\circ$

$2\pi/5 = 72^\circ$

etc.

Thus, the solutions of $z^5 = 1$ are

$z_1 = 1\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$

$z_2 = 1\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$

$z_3 = 1\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right)$

* $z_4 = 1\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right)$

$z_5 = 1\left(\cos\frac{10\pi}{5} + i\sin\frac{10\pi}{5}\right) = \cos 2\pi + i\sin 2\pi$

$= \cos 0 + i\sin 0 = 1$

Note that $z_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$

$$= \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-2\pi/5\right)$$

$$= \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} = \bar{z}_1$$

(the conjugate of z_1).

Similarly (check!), $z_3 = \bar{z}_2$.

We say that z_1, z_2, z_3, z_4, z_5 are the five fifth roots of 1 ('unity').

Theorem: If a is a root of $f(x)$ (ie satisfies $f(a)=0$) then $(x-a)$ is a factor of $f(x)$.

Conclusion: The linear factors of $x^5 - 1$ are

$(x-z_1), (x-z_2), (x-z_3), (x-z_4)$ and $(x-z_5)$.

ie $\quad x^5 - 1 = (x-z_1)(x-z_2)(x-z_3)(x-z_4)(x-z_5)$.