Lecture 12

Remember the result that led us to de Moivre's theorem:

that for any complex numbers z and ω,

(1) $|z\omega| = |z|\,|\omega|$

(2) $\operatorname{Arg}(z\omega) = \operatorname{Arg}(z) + \operatorname{Arg}(\omega)$.

It is worth noting that since $\frac{\omega}{\omega} = \omega \cdot \frac{1}{\omega} = 1$

then $|\omega \frac{1}{\omega}| = |\omega|\,|\omega^{-1}| = 1 \Rightarrow |\omega^{-1}| = \frac{1}{|\omega|}$

ie $\left|\frac{1}{\omega}\right| = \frac{1}{|\omega|}$

ie to calculate $|\omega^{-1}| = \left|\frac{1}{\omega}\right|$, we only need $|\omega|$ for then $\frac{1}{|\omega|}$ gives us $\left|\frac{1}{\omega}\right|$.

From this observation, it follows that

$$\left|\frac{z}{\omega}\right| = \left|z \cdot \frac{1}{\omega}\right| \underset{\text{from (1)}}{=} |z|\left|\frac{1}{\omega}\right| = |z|\frac{1}{|\omega|} = \frac{|z|}{|\omega|}$$

ie (3) $\left|\frac{z}{\omega}\right| = \frac{|z|}{|\omega|}$

Note that if $\omega = x+iy$, then $\frac{1}{\omega} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$

Conjugate of ω

remember: multiply by $\frac{x-iy}{x-iy}$

So $\frac{1}{\omega} = \frac{\overline{\omega}}{|\omega|^2}$ means that

$$\operatorname{Arg}\left(\frac{1}{\omega}\right) = -\operatorname{Arg}(\omega).$$

This also comes directly from de Moivre's theorem:

$$\operatorname{Arg}(\omega^n) = n\operatorname{Arg}(\omega)$$

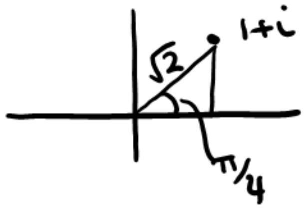
taking $n = -1$.

It follows that $\operatorname{Arg}\left(\frac{z}{\omega}\right) = \operatorname{Arg}(z\,\omega^{-1}) = \operatorname{Arg}(z) + \operatorname{Arg}(\omega^{-1})$

$$= \operatorname{Arg}(z) - \operatorname{Arg}(w)$$

ie (*) $\operatorname{Arg}\left(\frac{z}{w}\right) = \operatorname{Arg}(z) - \operatorname{Arg}(w).$

<u>Problem</u> Evaluate $\left(\frac{1+i}{1-i}\right)^{10}$.



<u>Solution</u> $1+i = \sqrt{2}\,(\cos \pi/4 + i \sin \pi/4)$

$1-i = \sqrt{2}\left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right)$

So $\left(\frac{1+i}{1-i}\right)^{10} = \left(\frac{\sqrt{2}\,(\cos \pi/4 + i\sin \pi/4)}{\sqrt{2}\,(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4})}\right)^{10} = \frac{(\cos \pi/4 + i \sin \pi/4)^{10}}{(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4})^{10}}$

$$= \frac{\cos \frac{10\pi}{4} + i \sin \frac{10\pi}{4}}{\cos \frac{70\pi}{4} + i \sin \frac{70\pi}{4}}$$

$$= \frac{\cos \pi/2 + i \sin \pi/2}{\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}}$$

$$= \cos(-\pi) + i \sin(-\pi)$$

$$= \cos(\pi) - i \sin(\pi)$$

$$= -1$$

Recall: for any θ,
$\cos(\theta) = \cos(-\theta)$
$\sin(-\theta) = -\sin(\theta)$.

<u>Problem</u> Find all z such that $z^4 = 1$.
Then factorize the polynomial $x^4 - 1$ as

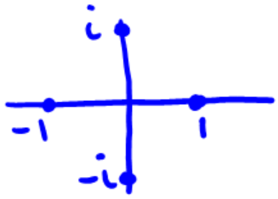
i) a product of linear factors

ii) a product of <u>real</u> linear/quadratic factors.

Solution: It's easy to see by inspection that

$1^4 = 1$, $(-1)^4 = 1$, $i^4 = 1$ (because $i^2 = -1$ and so $(i^2)^2 = (-1)^2 = 1$)

and $(-i)^4 = 1$.



Next $p(x) = x^4 - 1$

$p(1) = 0$, $p(-1) = 0$, $p(i) = 0$, $p(-i) = 0$

i) a product of linear factors

So $x^4 - 1 = \underbrace{(x-1)(x+1)(x-i)(x+i)}$

ii) a product of REAL factors $= (x-1)(x+1)(x^2+1)$.