Lecture 11

<u>Recall</u>: for any complex numbers z and ω,

we know that $\quad \operatorname{Arg}(z\omega) = \operatorname{Arg}(z) + \operatorname{Arg}(\omega)$

and $\quad |z\omega| = |z|.|\omega|$.

It follows that $\operatorname{Arg}(z^2) = \operatorname{Arg}(z\cdot z) = \operatorname{Arg}(z) + \operatorname{Arg}(z)$
$= 2\operatorname{Arg}(z)$.

So if $z = r(\cos\theta + i\sin\theta)$,

then $\operatorname{Arg}(z^2) = 2\theta$.

Further, $\operatorname{Arg}(z^3) = \operatorname{Arg}(z^2 . z) = \operatorname{Arg}(z^2) + \operatorname{Arg}(z)$
$= 2\theta + \theta = 3\theta$

$\vdots$

$\operatorname{Arg}(z^n) = n\theta$

ie $\quad \operatorname{Arg}(z^n) = n \operatorname{Arg}(z)$.

Finally, $\quad |z^2| = |z|.|z| = |z|^2$

$|z^3| = |z^2|.|z| = |z|^2.|z| = |z|^3$

$\vdots$

$|z^n| = |z|^n$.

<u>de Moivre's Theorem</u> extends the above result to <u>all</u> integers ($\mathbb{Z}$):

for $n \in \mathbb{Z}$, $\quad |z^n| = |z|^n$

and $\quad \operatorname{Arg}(z^n) = n\operatorname{Arg}(z)$.

Example : Evaluate $\left(\dfrac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$.

Let $\omega = 1+\sqrt{3}i$, $z = 1-\sqrt{3}i$.

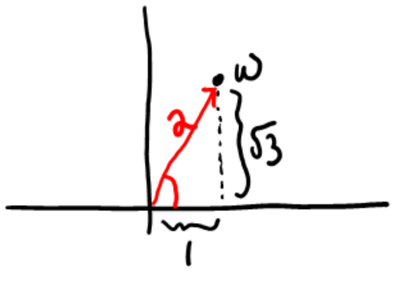
Need to calculate $\left(\dfrac{\omega}{z}\right)^{10}$

$= (\omega . z^{-1})^{10}$

$= \omega^{10} . z^{-10}$

(Could turn it into form $(x+iy)^{10}$ $\leftarrow$ $r = \sqrt{x^2+y^2}$, $\theta \ldots$

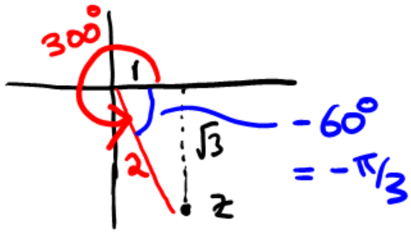
$z^{10} = r^{10}(\cos 10\theta + i\sin 10\theta)$)



$|\omega| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

$\arg(\omega) = 60^\circ = \pi/3$

$\omega^{10} = 2^{10}\left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right)$



$|z| = 2$, $\arg(z) = -\pi/3$

$\therefore \quad z^{-10} = 2^{-10}\left(\cos \frac{+10\pi}{3} + i \sin \frac{+10\pi}{3}\right)$

Finally, $w^{10} \cdot z^{-10} = 2^{10}\left(\cos \frac{10\pi}{3} + i \sin \frac{10\pi}{3}\right) \cdot 2^{-10}\left(\cos \frac{+10\pi}{3} + i \sin \frac{+10\pi}{3}\right)$

$\frac{1}{2^{10}}$

$= \cos \frac{20\pi}{3} + i \sin \frac{20\pi}{3}$

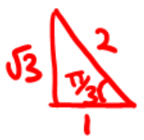
$\frac{10\pi}{3} + \frac{10\pi}{3}$ (applying de Moivre)

$= \cos \frac{(18+2)\pi}{3} + i \sin (18+2)\pi/3$

$= \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$

$= -\cos \pi/3 + i \sin \pi/3$

$= -\frac{1}{2} + \frac{\sqrt{3}}{2} i$



ie $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$