Lecture 10

Problem: Find $|z|$ and $\operatorname{Arg}(z)$ for $z = \dfrac{3i^{30} - i^{19}}{2i-1}$.

First, let's simplify the numerator:

$i^2 = -1$ so $i^{30} = (i^2)^{15} = (-1)^{15} = -1$

$$i^{19} = i \cdot i^{18} = i\,(i^2)^9 = i(-1)^9 = i(-1) = -i$$

So $$z = \frac{3(-1) - (-i)}{2i-1} = \frac{-3+i}{2i-1}$$

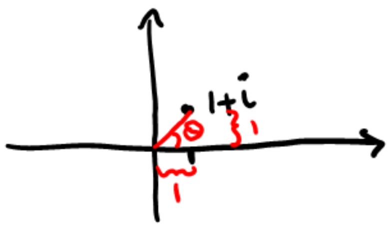
$$= \frac{-3+i}{-1+2i} \cdot \frac{(-1-2i)}{(-1-2i)}$$

Recall: $(a-b)(a+b) = a^2 - b^2$ where $a=-1,\ b=2i$

$$= \frac{3 + i(-2i) - 3(-2i) + i(-1)}{1+4}$$

$$= \frac{3+2+5i}{5} = \frac{5+5i}{5} = 1+i$$

Finally, we have z in the form of $x+iy$ where $x \in \mathbb{R},\ y \in \mathbb{R}$.



$\operatorname{Arg}(z) = \theta = \pi/4$

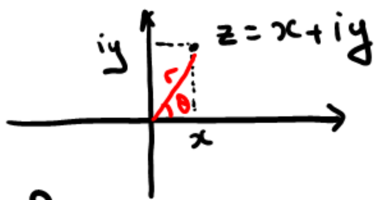
$|z| = \sqrt{1^2+1^2} = \sqrt{2}$.

Some theory will lesson the burden for many such calculations:

Let $z = x+iy$, where $x, y \in \mathbb{R}$, and $i^2=-1$.

Let $|z| = r = \sqrt{x^2+y^2}$

$\operatorname{Arg}(z) = \theta$.



Now $\cos\theta = x/r \Rightarrow x = r\cos\theta$

$\sin\theta = y/r \Rightarrow y = r\sin\theta$

So $z = x+iy = r\cos\theta + i r\sin\theta = r(\cos\theta + i\sin\theta)$

Theorem. For any complex numbers w, z,

$$|wz| = |w|\,|z| \quad \text{and}$$

$$\mathrm{Arg}(wz) = \mathrm{Arg}(w) + \mathrm{Arg}(z)$$

Consider what this is telling us about wz.

Proof Let $w = r(\cos\theta + i\sin\theta)$, $z = s(\cos\varphi + i\sin\varphi)$

(so $|w| = r$, $\mathrm{Arg}(w) = \theta$; $|z| = s$, $\mathrm{Arg}(z) = \varphi$.)

Then
$$\begin{aligned}
wz &= r(\cos\theta + i\sin\theta) . s(\cos\varphi + i\sin\varphi) \\
&= rs\,(\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi) \\
&= rs\,(\cos\theta\cos\varphi + i\cos\theta\sin\varphi + i\sin\theta\cos\varphi + i^2\sin\theta\sin\varphi) \\
&= rs\,(\cos\theta\cos\varphi - \sin\theta\sin\varphi + i(\cos\theta\sin\varphi + \sin\theta\cos\varphi)) \\
&= rs\,(\cos(\theta + \varphi) + i\sin(\theta + \varphi))
\end{aligned}$$

ie $|wz| = rs = |w||s|$

and $\mathrm{Arg}(wz) = \theta + \varphi = \mathrm{Arg}(w) + \mathrm{Arg}(z)$ $\square$

In particular, $\mathrm{Arg}(z^2) = \mathrm{Arg}(z.z) = \mathrm{Arg}(z) + \mathrm{Arg}(z)$

$= 2\,\mathrm{Arg}(z)$.