

MA133C & MA160

Calculus 1

Lecture 8



Limits of functions: examples

Example. Consider the function

$$f(x) = \frac{3x^2 - 6x + 3}{x - 1}.$$

We know from our study of rational functions that the natural domain of this function is $\mathbb{R} \setminus \{1\}$ and that the limits at $+\infty$ and $-\infty$ are not finite.

What happens near the point excluded from the domain? Or formally: what is

$$\lim_{x \rightarrow 1} \frac{3x^2 - 6x + 3}{x - 1}?$$

Limits of functions: examples

We cannot simply substitute for x the value 1, but we can evaluate f at values close to this point:

$$f(0.9) = -0.2999999999999998, \quad f(0.99) = -0.030000000000000189, \quad \dots,$$

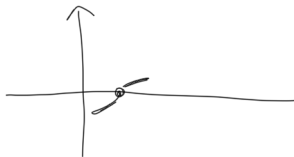
$$f(1.1) = 0.3000000000000002, \quad f(1.01) = 0.029999999999999301,$$

$$f(1.001) = 0.002999999999953149 \dots$$

We notice that for values of x close to 1 the values of f become arbitrarily close to 0.

Formally we will write:

$$\lim_{x \rightarrow 1} \frac{3x^2 - 6x + 3}{x - 1} = 0.$$



Limits of functions: definition

Limit at x_0 (finite limit)

Suppose $f(x)$ is defined for x close to x_0 . We write

$$\lim_{x \rightarrow x_0} f(x) = L$$

“the limit of $f(x)$, as x approaches x_0 , equals L ”

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to x_0 (on either side) but not equal to x_0 .

Limit at x_0 (infinite limit)

Suppose $f(x)$ is defined for x close to x_0 . We write

$$\lim_{x \rightarrow x_0} f(x) = +\infty \quad (\text{resp. } -\infty)$$

if we can make the values of $f(x)$ arbitrarily large (resp. arbitrarily large negative) taking x to be sufficiently close to x_0 (on either side) but not equal to x_0 .

Right and left limits and vertical asymptotes

If in the definition of the limit we allow x near x_0 but always $x > x_0$ what we get is the so-called **right** limit at x_0 :

$$\lim_{x \rightarrow x_0^+} f(x).$$

Similarly, if we only allow x near x_0 but always $x < x_0$ what we get is the so-called **left** limit at x_0 :

$$\lim_{x \rightarrow x_0^-} f(x).$$

Vertical asymptote

We say that the vertical line $x = x_0$ is a vertical asymptote for the function f if the limit at x_0 or at least the right or left limits at x_0 is $+\infty$ or $-\infty$.

Example The vertical line $x = 2$ is a vertical asymptote for $f(x) = \frac{x-1}{x-2}$.

Examples

Example

Let $f(x) = \frac{x-3}{x^2}$. Compute the limits at $x = 0, 1, 2, 3$.

$$\lim_{x \rightarrow 3} \frac{x-3}{x^2} = 0$$

$$f(3.1) = \frac{3.1-3}{(3.1)^2} = \frac{0.1}{(3.1)^2} \quad ; \quad f(3.01) = \frac{3.01-3}{(3.01)^2} = \frac{0.01}{\dots}$$

$$f(0.1) = \frac{-2.9}{0.01} \quad \leftarrow \text{Big}$$

going closer to 0 we get bigger values of f but negative

...

$$\lim_{x \rightarrow 0} f(x) = -\infty$$

Limits of functions: (helpful!) rules

Suppose that $\sqrt{c} \in \mathbb{R}$ and the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist. Then¹

- $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x)$
- $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ provided $\lim_{x \rightarrow a} g(x) \neq 0$

$$\frac{x-3}{x^2}$$
$$f(x) = x-3 \quad \lim_{x \rightarrow 3} f(x) = 0$$
$$g(x) = x^2 \quad \lim_{x \rightarrow 3} g(x) = 9$$

5.

$$\Rightarrow \lim_{x \rightarrow 3} \frac{x-3}{x^2} = 0$$

¹Sections 2.2 and 2.3 of Stewart contain a lot of instructive examples and more rules

Example

$$\lim_{x \rightarrow 1} \left[\overbrace{\frac{3x^2 - 6x + 3}{x - 1}}^f + \overbrace{2x(x^5 - 1)}^g \right] = \quad \checkmark \quad 1.$$

$$\lim_{x \rightarrow 1} \frac{3x^2 - 6x + 3}{x - 1} + \lim_{x \rightarrow 1} 2x(x^5 - 1) = \dots$$

$$= 0 + \left(\lim_{x \rightarrow 1} 2x \right) \cdot \left(\lim_{x \rightarrow 1} x^5 - 1 \right) = 0$$

from
1st slide

$$\begin{aligned} g(x) &= 2x(x^5 - 1) \\ (2x) \cdot (x^5 - 1) \\ \uparrow & \quad \uparrow \\ \lim_{x \rightarrow 1} 2x &= 2 \\ \lim_{x \rightarrow 1} x^5 - 1 &= 0 \end{aligned}$$

More limit laws

Implicitly, we used

- (a) If f is a polynomial, a rational or an algebraic function and a is in the domain of f then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

- (b) If $f(x) = g(x)$ when $x \neq a$ then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$ provided the limits exist.

Going back to our **initial example**: consider

$$f(x) = \frac{3x^2 - 6x + 3}{x - 1} \quad \text{and} \quad g(x) = 3x - 3$$

$$f(x) = \frac{3(x^2 - 2x + 1)}{x - 1} = \frac{3(x - 1)^2}{x - 1}$$

More examples

Let $g(x) = \frac{\sqrt{x^2+9}-3}{x^2}$. Compute the limits at $x = 0, 1, 2, 3$.

Domain of g is $\mathbb{R} - \{0\}$

Rule (a) tells us:

$$\lim_{x \rightarrow 3} g(x) = g(3) = \frac{3\sqrt{2}-3}{9} = \frac{\sqrt{2}-1}{3}$$

Similarly for $\lim_{x \rightarrow 1}$ & $\lim_{x \rightarrow 2}$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{x^2+9}-3}{x^2} &= \lim_{x \rightarrow 0} \frac{\sqrt{x^2+9}-3}{x^2} \cdot \frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3} = \lim_{x \rightarrow 0} \frac{\overset{\lim_{x \rightarrow 0}}{x^2+9-9}}{x^2(\sqrt{x^2+9}+3)} = \\ &= \lim_{x \rightarrow 0} \frac{\overset{(1)}{x^2}}{x^2(\sqrt{x^2+9}+3)} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x^2+9}+3} = \frac{1}{\sqrt{0+9}+3} = \frac{1}{6} \end{aligned}$$

More examples

Let $h(x) = \frac{x^2 + 5x + 4}{x^2 + 3x - 4}$. (1) Compute the limits at $x = -4, 0, 1$.

(2) Does h have vertical asymptotes?

Let us start by rewriting $h(x)$ in a convenient way:

$$h(x) = \frac{(x+4)(x+1)}{(x+4)(x-1)}$$

The domain of h is $\mathbb{R} \setminus \{-4, 1\}$

• by our rules, $\lim_{x \rightarrow 0} h(x)$ coincides with $h(0)$: $\lim_{x \rightarrow 0} h(x) = h(0) = \frac{4}{-4} = -1$

• to evaluate the lim at -4 & 1 we will use (b) from above and the function $g(x) = \frac{x+1}{x-1}$ which coincides with $h(x)$ outside these points.

$$\lim_{x \rightarrow -4} \frac{(x+4)(x+1)}{(x+4)(x-1)} = \lim_{x \rightarrow -4} \frac{x+1}{x-1} = \frac{-4+1}{-4-1} = \frac{3}{5}$$

Near 1 the function $g(x)$ changes sign. This tells us that the right and left limits might not coincide. (the function takes bigger and bigger values for x near 1 with $x > 1$)

$$\lim_{x \rightarrow 1^+} \frac{x+1}{x-1} = +\infty$$

$$\lim_{x \rightarrow 1^-} \frac{x+1}{x-1} = -\infty$$

so $x=1$ is a vertical asymptote for $h(x)$.