MA133C & MA160

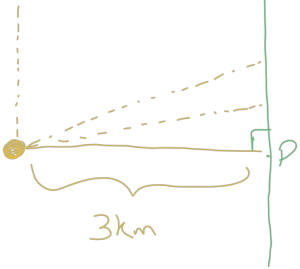
Calculus 1

Lecture 6

Trigonometric functions: an exercise

A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light beam makes one revolution every four minutes. The light beam passes point P at time $t = 0$ seconds.

1. Give a formula, valid for $0 \leqslant t < 60$, for the distance D (in km) between the point P and the point where the light beam hits the shore at time t seconds.
2. Evaluate $D(t)$ for $t = 0, 15, 30, 45$ seconds.
3. Sketch the graph of $D(t)$ for the time interval $0 \leqslant t \leqslant 45$.



$t=0 \longrightarrow \alpha = 0$

$t=240 \longrightarrow \alpha = 2\pi$

$\alpha = \dfrac{2\pi t}{240}$ at time t

$$D(t) = 3 \cdot \tan\left(\frac{2\pi t}{240}\right)$$

$t = 0, 15, 30, 45$

$D(0) = 3 \cdot 0 =$

$D(15) = 3 \cdot \tan\left(\frac{2\pi \cdot 15}{240}\right) = 3 \cdot \tan\left(\frac{\pi}{8}\right)$

$D(30) = 3 \cdot \tan\left(\frac{\pi}{4}\right) = 3$

$D(45) = 3 \cdot \tan\left(\frac{\overset{3}{\cancel{90}}}{\underset{8}{\cancel{240}}}\pi\right) = 3 \cdot \tan\left(\frac{3\pi}{8}\right)$

New functions from old: composition

To define new functions from old, apart from the usual algebraic operations we have often implicitly used the **composition of functions**. Let's have a closer look at this operation.

In general, given any two functions f and g, we start with a number x in the domain of g and compute its image $g(x)$.

If this number $g(x)$ is in the domain of f, then we can calculate the value of f at $g(x)$: $f(g(x))$.

The result is a new function, called the **composition** of f and g, denoted $f \circ g$. Formally:

Definition: composition of functions

Given two functions f and g, the composition of f and g is defined by

$$(f \circ g)(x) = f(g(x)).$$

⚠ The order matters! See the following examples.

Examples

for $f \circ g$ we substitute $\frac{1+x}{x}$ in place of x in $f(x)$.

(1) Let $f(x) = \sqrt{x} - 1$ and $g(x) = \frac{1+x}{x}$. Write an expression for $f \circ g$ and an expression for $g \circ f$. What are the domains of the new functions? Write coordinates for the x- and y-intercepts of each of the new functions (if they exist).

$$(f \circ g)(x) = \sqrt{\frac{1+x}{x}} - 1$$

Domain: $x \neq 0$; $\frac{1+x}{x} \geq 0$

$1 + x \geq 0 \quad x \geq -1$

$x \geq 0$

	-1		0	
N	$-$	$+$		$+$
D	$-$	$-$		$+$
	$+$	$-$		$+$

Domain of $f \circ g$ is: $(-\infty, -1] \cup (0, +\infty)$

No y-axis intercept

x-intercept:

$\sqrt{\frac{1+x}{x}} - 1 = 0 \qquad \sqrt{1+x} = \sqrt{x}$

$\cancel{x} + 1 = \cancel{x}$

no solution

(see the last page for $g \circ f$..)

Examples

(2) Let $f(x) = \cos x$ and $g(x) = 1 - x^2$. Write an expression for $f \circ g$ and an expression for $g \circ f$. What are the domains of the new functions?

As both f and g are defined on $\mathbb{R}$, the compositions $f \circ g$ and $g \circ f$ will have domain $\mathbb{R}$.

$$(f \circ g)(x) = \cos(1 - x^2)$$

$$(g \circ f)(x) = 1 - (\cos x)^2 = (\sin x)^2$$

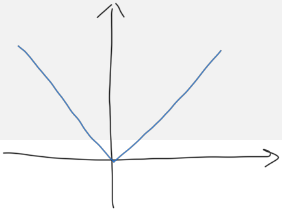
Absolute value

Sometimes functions are defined through different expressions in different portions of their domains. We call this kind of functions **piecewise defined functions**.
One important example of functions of this type is the **absolute value**.

Definition: absolute value

The absolute value of x is denoted $|x|$ and it is defined as

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0. \end{cases}$$



Examples

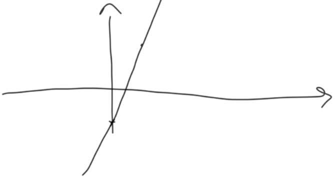
1. $|123| = 123$ and $|-\pi| = \pi$.
2. The solutions to $|x-3| = 2$ are: $x = 1$ and $x = 5$.
3. All real numbers satisfy $1 - |\sin(x)| \geqslant 0$.

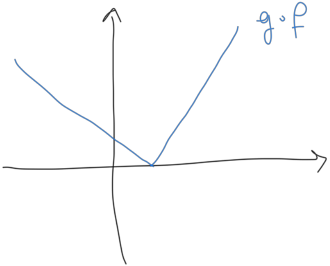
Examples

Let $f(x) = 3x - 1$ and $g(x) = |x|$. Write an expression for $f \circ g$ and an expression for $g \circ f$. Write coordinates for the x- and y-intercepts of each of the new functions and sketch their graphs.

$(g \circ f)(x) = |3x - 1|$

(consider $h(x) = 3x - 1$ first)

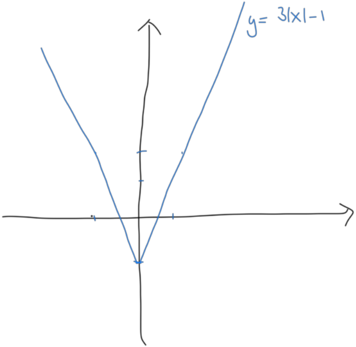




Examples

Let $f(x) = 3x - 1$ and $g(x) = |x|$. Write an expression for $f \circ g$ and an expression for $g \circ f$. Write coordinates for the x- and y-intercepts of each of the new functions and sketch their graphs.

$$(f \circ g)(x) = 3|x| - 1 = \begin{cases} 3x - 1 & \text{for } x \geq 0 \\ -3x - 1 & \text{for } x < 0 \end{cases}$$



$f(g(0)) = -1$

$f(g(1)) = 3 \cdot 1 - 1 = 2$

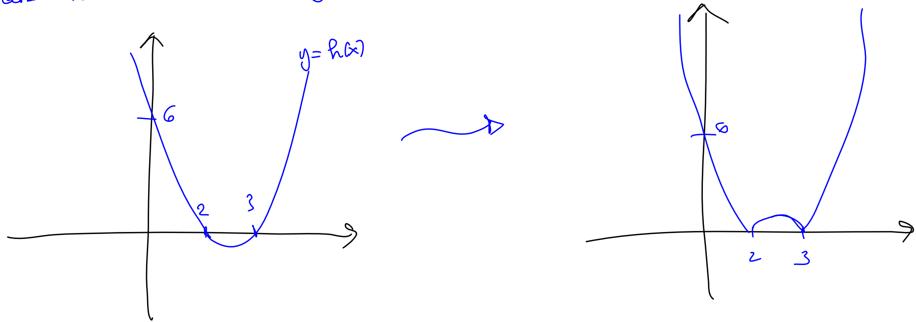
$f(g(-1)) = (-3)(-1) - 1 = 2$

Examples

Sketch the graph of $f(x) = |x^2 - 5x + 6|$.

Consider the quadratic function $h(x) = x^2 - 5x + 6$. Its graph is a parabola facing up, with x-axis intercepts $x = 3$, $x = 2$,
· y-axis intercept $y = 6$

To get the graph of f from that of h, simply "fold" the Cartesian plane along $y = 0$, and draw the mirror image of what's below the x-axis!



Examples

From example (1):

$f(x)=\sqrt{x}-1 \qquad g(x)=\dfrac{1+x}{x} \qquad$ then $(g\circ f)(x)=\dfrac{1+\left(\sqrt{x}-1\right)}{\sqrt{x}-1}=\dfrac{\sqrt{x}}{\sqrt{x}-1}$

Domain: $x\geq 0$ and $\sqrt{x}-1\neq 0$, that is $x=1$, therefore: $g\circ f$ has domain $[0,1)\cup(1,+\infty)$

$(g\circ f)(0)=0$ is the only axis intercept.