

MA133C & MA160

Calculus 1

Lecture 3



Recap

- ▶ Polynomials are functions of the form:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers a_0, \dots, a_n are constants. If $a_n \neq 0$ then n is called the degree of the polynomial.

- ▶ When $n = 2$ the polynomial function is a quadratic function $f(x) = ax^2 + bx + c$. Its graph is a parabola. To sketch the graph: keep in mind the shape of the parabola $y = ax^2$ and shift it in the Cartesian plane according to its “vertex” and the axis intercepts.
- ▶ Similar reasoning for polynomials of higher degree, keeping in mind the shape of powers of even degrees (similar to a parabola) and those of odd degrees (similar to a cubic function).

On root and reciprocal functions

★ **Case 2:** $a = 1/n$ where n is a positive integer.

The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**.

When n is even, the root function $f(x) = \sqrt[n]{x}$ is only defined for nonnegative real numbers, that is its natural domain is $\mathbb{R}_{>0}$.

When n is odd, $f(x) = \sqrt[n]{x}$ is defined for all real numbers.

★ **Case 3:** $a = -1$.

We call the function $f(x) = x^{-1} = 1/x$ the **reciprocal function**, with natural domain $\mathbb{R} \setminus \{0\}$. The graph of the function $f(x) = 1/x$ is a hyperbola.

Rational and algebraic functions

The reciprocal functional is a very special case of rational function.

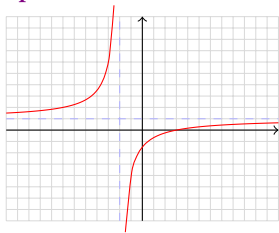
Rational functions

A **rational function** is a ratio of two polynomial functions p and q :

$$f(x) = \frac{p(x)}{q(x)}.$$

The natural domain of such function is the set of all real numbers for which $q(x) \neq 0$.

Example 1.



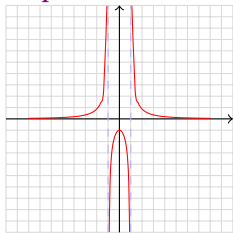
$$f(x) = \frac{x-3}{x+2} = \frac{(x+2)-5}{x+2} = 1 - \frac{5}{x+2}$$

This function is defined on $\{x \in \mathbb{R} : x \neq -2\} = \mathbb{R} \setminus \{-2\}$.

Its graph can be obtained by that of the reciprocal function.

Rational and algebraic functions

Example 2.

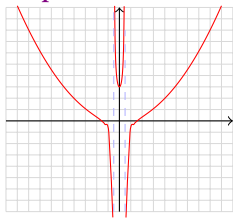


$$f(x) = \frac{1 + 3x^2}{x^4 - 1} = \frac{1 + 3x^2}{(x^2 - 1)(x^2 + 1)} = \frac{1 + 3x^2}{(x - 1)(x + 1)(x^2 + 1)}$$

This function is defined on $\{x \in \mathbb{R} : x \neq \pm 1\} = \mathbb{R} \setminus \{-1, 1\}$.

Note that for very big and for very small values of x the values of the function are very close to 0.

Example 3.



$$f(x) = \frac{x^4 - 6}{8x^2 - 2} = \frac{x^4 - 6}{2(2x - 1)(2x + 1)}$$

This function is defined on $\{x \in \mathbb{R} : x \neq \pm \frac{1}{2}\} = \mathbb{R} \setminus \{-\frac{1}{2}, \frac{1}{2}\}$

What happens here when “ x goes to $+\infty$ or $-\infty$ ”?

How do you explain the symmetry with respect to the y -axis in these two examples?

Rational and algebraic functions

More generally, we can combine the various types of functions that we have seen so far to produce new functions.

Algebraic functions

An **algebraic function** is a function that can be constructed using algebraic operations (such as addition, subtraction, multiplication, division and taking roots) starting with polynomials.

Graphs of algebraic functions can assume various different shapes, and in most cases we will need more advanced tools to study their behaviour.

Some examples

$$f(x) = \sqrt{x^2 + 1} + \frac{2x + 1}{x + 4}; \quad g(x) = \frac{1 + x}{1 - \sqrt{x}}; \quad h(x) = \sqrt[3]{x} \cdot \frac{1 - x - x^2}{x^4 + 5}.$$

Can you tell what is the natural domain of each of them?

$$\bullet f(x) = \sqrt{x^2+1} + \frac{2x+1}{x+4}$$

• the rational function $\frac{2x+1}{x+4}$ is defined

for $x \neq -4$

• the square root is defined when its argument is nonnegative:

$x^2+1 \geq 0$ which holds for all $x \in \mathbb{R}$

\Rightarrow the natural domain of f is $\mathbb{R} - \{-4\}$.

$$\bullet g(x) = \frac{1+x}{1-\sqrt{x}}$$

g is defined where \sqrt{x} is defined AND the denominator is nonzero:

$$\rightarrow x \geq 0$$

AND

$$\rightarrow \sqrt{x} \neq 1 \Leftrightarrow x \neq 1$$

\Rightarrow the natural domain of g is $[0, 1) \cup (1, +\infty)$

$$\bullet h(x) = \sqrt[3]{x} \cdot \frac{1-x-x^2}{x^4+5}$$

is defined on all \mathbb{R} . Why?