MA133C & MA160
Calculus 1

Lecture 2

Recap

- A **function** is a rule that assigns to each element x in X exactly one element of a set Y, called $f(x)$.
  We call X the **domain** of the function f. The number $f(x)$ is called value of f at x.
  The set of possible values of $f(x)$ as x varies throughout the domain is called **range** of f.
- We will consider real-valued functions of one real variable, that is functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$.
- We can visualise a function by plotting its graph on the Cartesian plane. The graph will consist of points of the form $(x, f(x))$.
- A **linear function** is a function of the form $f(x) = ax + b$, for some real numbers a and b.

Linear functions: exercise

Exercise.

(a) As dry air moves upward, it expands and cools. If the ground temperature is 20^oC and the temperature at a height of $1km$ is 10^oC, express the temperature T (in ^oC) as a function of the height h, assuming that a linear model is appropriate.

(b) Draw the graph of the function in part (a). What does the slope represent?

(c) What is the temperature at a height of $2.5km$?

More functions on the catalog: Polynomials

Linear functions belong to a bigger family of functions: **polynomials**.

Definition

A **polynomial** is a function $p\colon \mathbb{R} \longrightarrow \mathbb{R}$ of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, \ldots, a_n$ are constants called the **coefficients** of the polynomial.

The *natural* domain of any polymial is $\mathbb{R}$. If the **leading coefficient** a_n is nonzero, then the **degree** of the polynomial is n.

Exmples:

- A linear function $f(x) = ax + b$ with $a \neq 0$ is a polynomial of degree 1. A constant function is a polynomial of degree 0.
- The function

$$p(x) = 6x^5 + 0.3x^3 - 2x^2 + 1$$

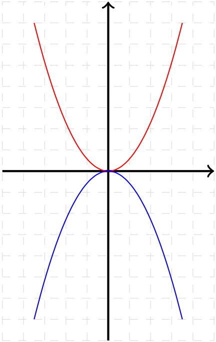
  is a polynomial of degree 5.

Quadratic functions and their graphs

A polynomial of degree 2 is called a **quadratic function**. It has this shape

$$f(x) = ax^2 + bx + c,$$

for $a \neq 0$, b and c fixed constants. The graph of a quadratic function is a **parabola**. It can be obtained by shifting the graph of the function $f(x) = ax^2$, which is



if $a > 0$ then the parabola opens upward

if $a < 0$ then the parabola opens downward

Any quadratic function

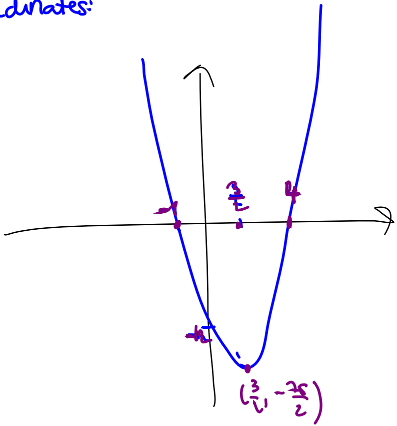
We can draw the graph of a quadratic function $f(x)=ax^2+bx+c$
by shifting the graph of a "basic" parabola:
If $f(x)=ax^2+bx+c$ then the top (or bottom, according to
whether $a>0$ or $a<0$) point has coordinates:

$$x=-\frac{b}{2a}, \quad y=f\left(-\frac{b}{2a}\right)$$

Example $f(x)=3x^2-9x-12$

$f(0)=-12$; $f(x)=0 \Leftrightarrow x=4$ or $x=-1$

min at $x=\frac{9}{6}=\frac{3}{2}$, $y=f\left(\frac{3}{2}\right)=-\frac{75}{4}$



Examples of quadratic functions

Models of motion

Quadratic functions are used in models involving motion of falling/thrown objects. The height of a falling object can be expressed as functions of time as follows:

$$h(t) = -4.9t^2 + v_0 t + h_0,$$

where v_0 is the initial velocity (expressed in m/s) and h_0 is the initial height (expressed in m).

Exercise. A ball is thrown upward from an initial height of $50m$ with an initial velocity of $20m/s$.

$\star$ Give the function that describes the height of the ball in terms of the time t.

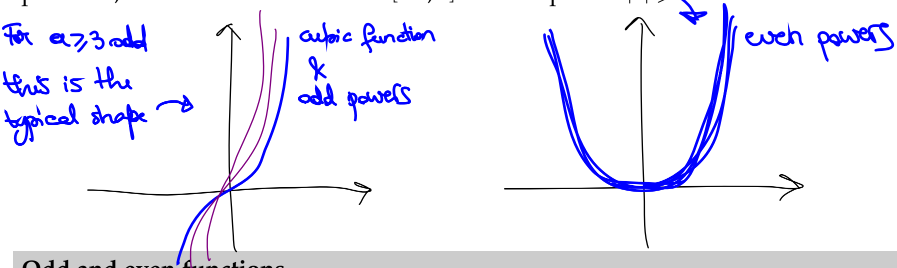
$\star$ After how many seconds does the ball reach its maximum height? What is this maximum height?

$\star$ After how many seconds will the ball hit the ground?

Powers

A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**.

⋆ **Case 1:** a is a positive integer. Then for $a \geqslant 2$ even the graph looks like the one of the parabola, but flatter in the interval $[-1, 1]$ and steeper for $|x| \geqslant 1$.



Odd and even functions

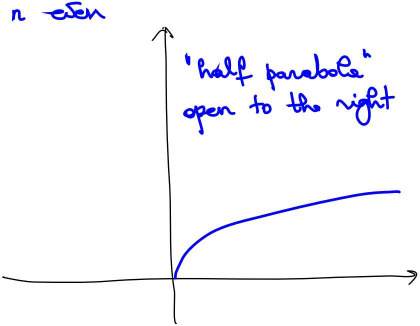
An **even** function f is a function satisfying $f(x) = f(-x)$ for all x in the domain.

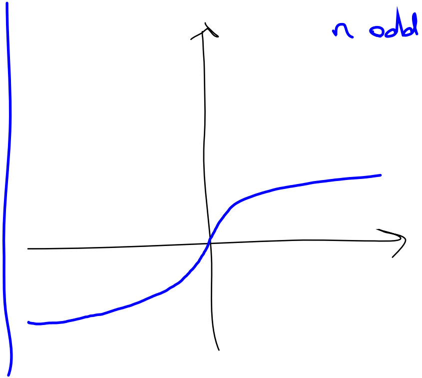
An **odd** function f is a function satisfying $f(-x) = -f(x)$ for all x in the domain.

The advantage of even and odd functions lies in their symmetry: their graphs are symmetric with respect to the y-axis (even functions) or with respect to the origin (odd functions). Even (resp. odd) powers are classical examples of even (resp. odd) functions.

⋆ **Case 2:** $a = 1/n$ where n is a positive integer.

The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**.
For all even n the natural domain of the root function is the set of all nonnegative real numbers $[0, \infty)$ and the graph is a half parabola open to the right.
For odd n the root function $f(x) = x^{1/n}$ is defined on all $\mathbb{R}$ and its graph can be obtained from that of the function x^n.

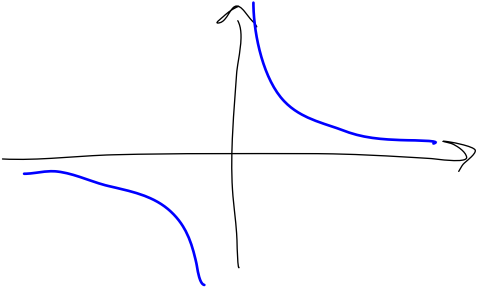




Powers

⋆ **Case 3:** $a = -1$.

We call the function $f(x) = x^{-1} = 1/x$ the **reciprocal function**. This function arises when one quantity is inverse proportional to another. This also means that their product is constant. The graph of the function $f(x) = 1/x$ is a hyperbola.



Next week: more "basic functions" and how to get new functions (and their graphs) from known basic ones.