Some more Practice on Multiplying matrices :

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$$

$$AB = \begin{pmatrix} (1\ 2)\begin{pmatrix}2\\1\end{pmatrix} & (1\ 2)\begin{pmatrix}-1\\3\end{pmatrix} \\ (3\ 4)\begin{pmatrix}2\\1\end{pmatrix} & (3\ 4)\begin{pmatrix}-1\\3\end{pmatrix} \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 10 & 9 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 7 \\ -2 & 28 \end{pmatrix}$$ Note $AB \neq BA$.

$$A^{-1} = \frac{1}{|A|} A^{*} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

$(\ |A| = (1)(4) - (3)(2)$ $$= \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

Check $A^{-1}A = AA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$$\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

<u>Application</u>: Solve The eqns

$$x + 2y = 5$$
$$3x + 4y = 11$$

$\Longleftrightarrow AX = b = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \end{pmatrix}$

$$AX = b \Rightarrow \underbrace{A^{-1}A}X = A^{-1}b$$
$$\Rightarrow IX = X = A^{-1}b = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}\begin{pmatrix} 5 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\therefore X = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ ie $x = 1, y = 2$ check.

Geometrically This says that the lines

$$x + 2y = 5 \quad \& \quad 3x + 4y = 11$$

intersect at the point $(1, 2)$

GETTING AHEAD OF OUR selves on Homework 2:

we are asked to Find AB where

$$A = \begin{pmatrix} 3 & 3 & 4 \\ 3 & 4 & 6 \\ 5 & 6 & 7 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 8 & -3 & -2 \\ -9 & -1 & 6 \\ 2 & 3 & -3 \end{pmatrix}$$

<u>Ans</u>: $$AB = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 5I.$$

$$\therefore (A)\left(\tfrac{1}{5}B\right) = \tfrac{1}{5}AB = \tfrac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so $$\tfrac{1}{5}(B) = \begin{pmatrix} 8/5 & -3/5 & -2/5 \\ -9/5 & -1/5 & 6/5 \\ 2/5 & 3/5 & -3/5 \end{pmatrix} = A^{-1}$$

& Similarly $$\tfrac{1}{5}A = B^{-1}.$$

§ Finding The image of a line ℓ under a linear Transformation with matrix A. <u>Ex</u>: Let $T((x,y)) = (-3x+2y, 4x+y)$

Then T has matrix $A = \begin{pmatrix} -3 & 2 \\ 4 & 1 \end{pmatrix}$

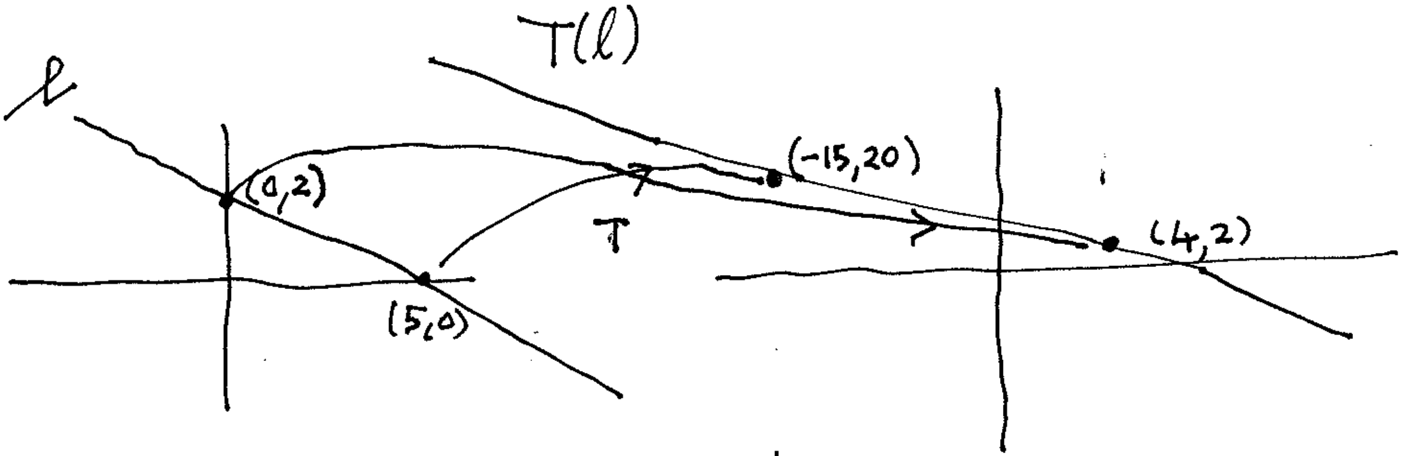
<u>Recall</u> To find the image under T of a point e.g. $(1, 2)$ we do:

$$\begin{pmatrix} -3 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \quad \text{so } T((1,2)) = (1,6)$$

To Find the image of a line ℓ under T just need to find the image of any 2 pts on ℓ & take the line through the 2 image points:

Ex: $\ell := \quad 2x + 5y = 10$

Clearly $(5,0)$ & $(0,2)$ lie on ℓ.



$$\begin{pmatrix} -3 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} -15 \\ 20 \end{pmatrix} := (x_1, y_1)$$

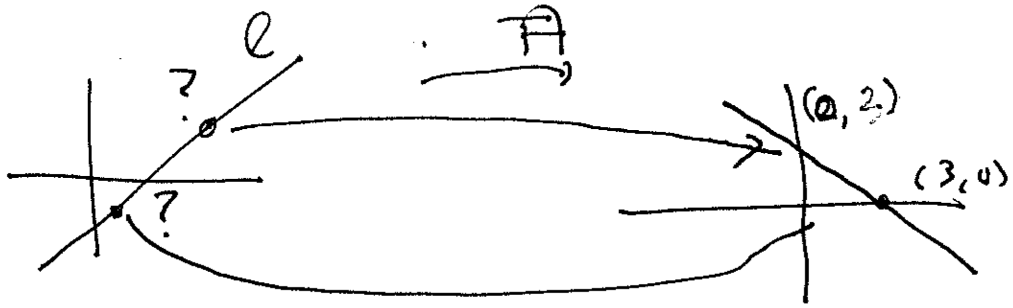
$$\begin{pmatrix} -3 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} := (x_2, y_2)$$

The eqn for the line through 2 pts (x_1, y_1) & (x_2, y_2) (hs books) is $y - y_1 = \underbrace{\dfrac{y_2 - y_1}{x_2 - x_1}}_{\text{slope}}(x - x_1)$

$$\therefore \quad y - 20 = \frac{-18}{19}(x + 15)$$

$$\Rightarrow \quad 18x + 19y = 110$$

Ex: Find the line whose image under T is the line $2x + 3y = 6$



Need to find the Points that get mapped to $(0,2)$ & $(3,0)$

clearly $A^{-1}\begin{pmatrix}0\\2\end{pmatrix} \xrightarrow[A]{} \begin{pmatrix}0\\2\end{pmatrix}$

Because $AA^{-1}\begin{pmatrix}0\\2\end{pmatrix} = I\begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$

& $A^{-1}\begin{pmatrix}3\\0\end{pmatrix} \xrightarrow[A]{} \begin{pmatrix}3\\0\end{pmatrix}$

Now $A^{-1} = -\frac{1}{11}\begin{pmatrix}1 & -2\\-4 & 3\end{pmatrix}$

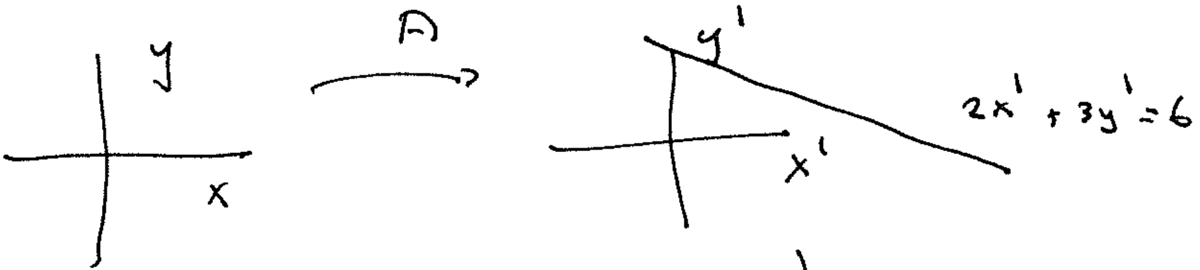
So $A^{-1}\begin{pmatrix}0\\2\end{pmatrix} = -\frac{1}{11}\begin{pmatrix}1 & -2\\-4 & 3\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} = -\frac{1}{11}\begin{pmatrix}-4\\-6\end{pmatrix}$

& $A^{-1}\begin{pmatrix}3\\0\end{pmatrix} = -\frac{1}{11}\begin{pmatrix}3\\-12\end{pmatrix}$

$\therefore\ y - 6/11 = -\frac{6}{7}(x - 4/11)$

$\Rightarrow\ 6x + 7y = 6$ gets mapped to $2x+3y=6$

An alternative method:



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3x + 2y \\ 4x + y \end{pmatrix}$$

So the line in the (x, y) plane whose image is $2x' + 3y' = 6$ is $2(-3x + 2y) + 3(4x + y) = 6$

$$\Rightarrow 6x + 7y = 6$$

Properties of Determinants:

(i) $|A| \neq 0 \iff A^{-1}$ exists

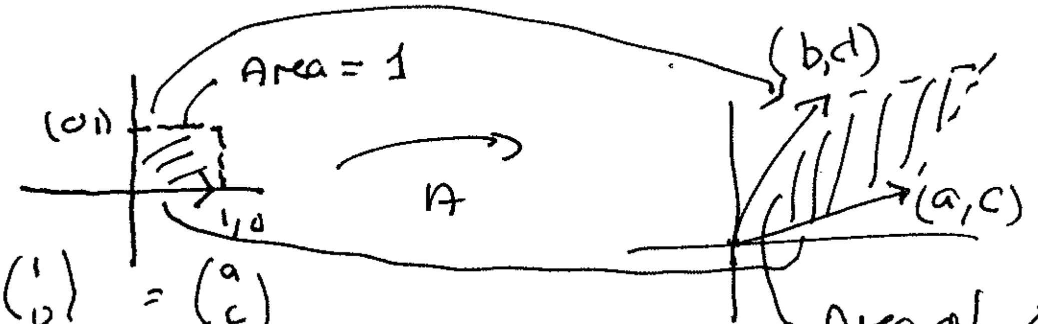
(ii) $|AB| = |A||B|$

(iii) $|A^{-1}A| = |I| = \left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 1$

$\quad \| $

$|A^{-1}||A|$ (By (ii)) $\Rightarrow |A^{-1}| = \dfrac{1}{|A|}$

(iv)



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

Area of ▱ $= \pm(ad - bc) = \pm|A|$

So $|A|$ = change of Area under the transformation A.