

Recall $A = \begin{pmatrix} 2 & -3 \\ 0 & 5 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix}$

$$AB = \begin{pmatrix} 5 & -11 \\ -5 & 25 \end{pmatrix} \quad BA = \begin{pmatrix} 12 & 2 \\ -15 & 5 \end{pmatrix} \\ = \begin{pmatrix} 2 & 7 \\ -2 & 25 \end{pmatrix}$$

$$AB \neq BA$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad AI = IA = A.$$

\forall matrix
A.

Ex: $L(x, y) = (ax + by, cx + dy)$

$$L \longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

The image of $v = (x, y)$ under L is obtained as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Rewrite as $(ax + by, cx + dy)$

Ex: Find the image of the vector $(2, 1)$ under the linear transformation

$$L(x, y) = (2x - 3y, 5y)$$



$$\begin{pmatrix} 2 & -3 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

So $(1, 5)$ is the image of $(2, 1)$

§ Systems of Linear Equations:

Ex: Solve

$$\begin{aligned} 2x + y &= 3 \\ x + 4y &= 2 \end{aligned} \quad (+)$$

Rewrite in terms of matrices:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} X = \begin{pmatrix} x \\ y \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

(+) becomes: $AX = b$

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

How to Find A^{-1} :

$$A^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

(note): $\begin{pmatrix} -c & a \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = -ac + ac = 0$

$$A^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(adjoint
of A)

$$|A| := ad - bc$$

The determinant
of A .

So

$$A^{-1} := \frac{1}{|A|} A^*$$

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$

$$|A| = (2)(4) - (1)(1) = 7$$

$$A^{-1} = \frac{1}{|A|} A^* = \frac{1}{7} \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$$

(check: $\frac{1}{7} \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$
 $\underbrace{\hspace{10em}}_{A^{-1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓)

So To Solve: $2x + y = 3$ $b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$
 $x + 4y = 2$

Rewrite:
 $X = \begin{pmatrix} x \\ y \end{pmatrix}$ $AX = b$ $A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$

$$A^{-1}AX = A^{-1}b$$

$$IX = A^{-1}b$$

$$\textcircled{6} \quad X = A^{-1}b$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 10 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{10}{7} \\ \frac{1}{7} \end{pmatrix}$$

$$x = \frac{10}{7} \quad y = \frac{1}{7}$$

Check: $2 \left(\frac{10}{7} \right) + \frac{1}{7} = \frac{21}{7} = 3 \checkmark$

$$\frac{10}{7} + \frac{4}{7} = \frac{14}{7} = 2 \checkmark$$

FACT: A has an inverse if and only if $|A| \neq 0$.