Theorem Let M be a $p \times p$ real symmetric matrix. For column vectors $v \in \mathbb{R}^p$ define

$$f : \mathbb{R}^p \to \mathbb{R}, \quad v \longmapsto f(v) = v^t M v.$$

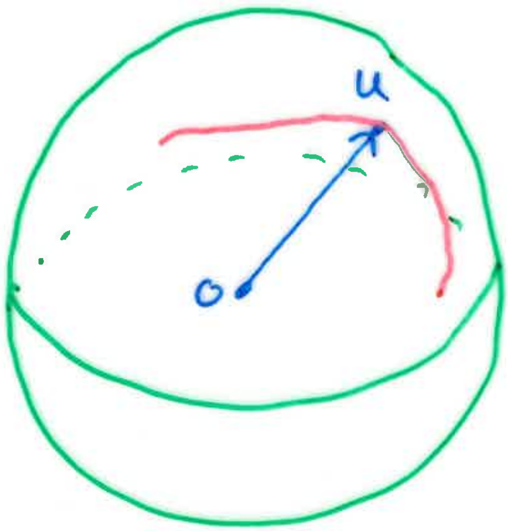
Let u be a point on the unit sphere

$$S^{p-1} = \{ v \in \mathbb{R}^p : \|v\| = 1 \}$$

for which $f(u)$ is a maximum for f on the sphere. Then

$$Mu = \lambda u$$

for some $\lambda \in \mathbb{R}$. (i.e. u is an eigenvector of M with eigenvalue λ.)



S^{p-1}

Proof Let $\mathcal{U} = \langle u \rangle$ be the subspace of $\mathbb{R}^p$ spanned by u.

Notation: for $w \in \mathbb{R}^p$ we'll write

$$u \cdot v = u^t v, \quad \|w\| = \sqrt{w^t w}.$$

Let

$$W = \{w \in \mathbb{R}^p : u \cdot w = 0\}$$

be the <u>subspace</u> of vectors orthogonal to u.

Then $\dim(\mathcal{U}) = 1$, $\dim(W) = p-1$

For any vector $w \in W \cap S^{p-1}$ define the curve

$$c(t) = (\cos t)u + (\sin t)w.$$

The curve $c(t)$ lies on the sphere S^{p-1} and passes through u. To see this

$$\|c(t)\| = \big((\cos t)u^t + (\sin t)w^t\big)\big((\cos t)u + (\sin t)w\big)$$

$$= (\cos^2 t)u^t u + (\sin^2 t)w^t w$$

$$= \cos^2 t + \sin^2 t$$

$$= 1.$$

Also $c(0) = u$

$$c'(t) = (-\sin t)\, u + (\cos t)\, w .$$

Thus, at u (i.e. when $t=0$), the direction of the curve is w and the curve is thus $\perp^r$ to u.

Consider

$$g(t) = f(c(t)) = c(t) . M c(t)$$

Since $f(u)$ is a maximum, and since $g(0) = f(u)$, it follows that $g'(0) = 0$.

Now

$$g'(t) = c'(t) . M c(t) + c(t) . M c'(t)$$

Aside $\quad w . M v = w^t M v$

$$v . M w = v^t M w = (w^t M v)^t$$

Thus $w . M v = v . M w$

Hence

$$g'(t) = 2\, c'(t) . M c(t)$$

$0 = g'(0) = 2\, c'(0) \cdot M c(0)$

$= 2\, w \cdot M u$

So Mu is perpendicular to w

for any $w \in W$.

This means $Mu = \lambda u$ with

$\lambda \in \mathbb{R}$. $\boxed{AT}$

Any linear homomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$

can be represented as

$$\phi(v) = M v$$

with M some $n \times n$ matrix (w.r.t. the standard basis)

A linear homomorphism is

symmetric

$$(\phi u) \cdot v = u \cdot (\phi v)$$

for all $u, v \in \mathbb{R}^n$. Note that ϕ is symmetric iff the matrix M is symmetric.

The above theorem can be restated as:

Theorem Any symmetric linear homomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ has an eigenvector.

Spectral Theorem Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a symmetric linear homomorphism. Then $\mathbb{R}^n$ has a basis consisting of eigenvectors of ϕ.

Proof Clearly ϕ has at least one eigenvector v say. Let $V = \langle v \rangle$ be spanned by v.

Let $V^\perp = \{ w \in \mathbb{R}^n : w \cdot v = 0 \}$.

ϕ restricts to homomorphisms

i) $\phi : V \longrightarrow V, \quad w \longmapsto \phi(w)$

ii) $\phi : V^\perp \longrightarrow V^\perp, \quad w \longmapsto \phi(w)$

for (i) we note for $w = rv$, $r \in \mathbb{R}$

$$\phi(w) = \phi(rv) = r\phi(v) = r\lambda v \in V$$

for (ii)

$$\begin{aligned} \phi(w) \cdot v &= w \cdot \phi(v) \\ &= w \cdot \lambda v \\ &= \lambda(w \cdot v) \\ &= 0 \end{aligned}$$

So $\phi : V^\perp \longrightarrow V^\perp$ can be regarded (!)

as a symmetric linear hom$^{\text{sm}}$

$$\phi : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}$$

and, as an inductive hypothesis,

we can assume that $V^\perp \cong \mathbb{R}^{n-1}$

has a basis consisting of $n-1$

eigenvectors of ϕ. By induction,

$\mathbb{R}^n$ has a basis of n eigenvectors

of ϕ. $\boxtimes$