Given $v_1, v_2, \dots, v_n \in \mathbb{R}^p$, $\frac{1}{n}(v_1 + \dots + v_n) = 0$

We set

$$X = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad C = \frac{1}{n} X X^t$$

find A, $AA^t = I$ so that the transformed data

$$A v_k = \begin{pmatrix} y_{k1} \\ \vdots \\ y_{kp} \end{pmatrix} \qquad 1 \le k \le n$$

will have covariance matrix

$$C' = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_p \end{pmatrix} \quad \text{with}$$

$$c'_{ii} = \lambda_i = \frac{1}{n} \sum_{k=1}^{n} y_{ki}^2 \qquad (**)$$

$$c'_{ij} = 0 \text{ for } i \ne j.$$

If $\lambda_i = 0$ for some i, then from $(**)$

$y_{ki} = 0$ for all $1 \le k \le n$.

Suppose $\lambda_{q+1} = \lambda_{q+2} = \dots = \lambda_p = 0$

for some $q \le p$.

Let B be the $q \times p$ matrix consisting of the first q rows of A. Then the linear map

$$\mathbb{R}^p \longrightarrow \mathbb{R}^q, \quad v \longmapsto Bv$$

is distance preserving on the set $\{v_1, \dots, v_n\}$.

i.e. $d(v_i, v_j) = d(Bv_i, Bv_j)$

for $1 \leq i, j \leq n$.

So the <u>geometry</u> of the data $v_1, v_2, \dots, v_n \in \mathbb{R}^p$ is the same as the geometry of the data $Bv_1, Bv_2, \dots, Bv_n \in \mathbb{R}^q$

In practice, we just require $\lambda_{q+1}, \dots, \lambda_p$ to be small but not necessarily zero.

Then $Bv_1, \dots, Bv_n$ have approximately the same geometry as $v_1, \dots, v_n$.

Example Gait Analysis

24 sensors place on a human.

Each sensor measures 3 Euler angles. At a given time t the sensors are represented by a vector $v_t \in \mathbb{R}^{87}$. The human is asked to i) walk forward along a track, ii) hop forward along a track.

In each case the time series of vectors $v_{t_1}, v_{t_2}, \dots, v_{t_n} \in \mathbb{R}^{87}$ can be viewed using PCA and the projection

$$\mathbb{R}^{87} \longrightarrow \mathbb{R}^3, \quad v \longmapsto Bv.$$

We need:

<u>Theorem</u> Let M be a $p \times p$ real symmetric matrix. For column vectors $v \in \mathbb{R}^p$ define

$$f : \mathbb{R}^p \to \mathbb{R}, \quad v \mapsto f(v) := v^t M v$$

Let u be a point on the unit sphere

$$S^{p-1} = \{ v \in \mathbb{R}^p : \|v\| = 1 \}$$

for which $f(u)$ is a maximum for f on the sphere. Then

$$M u = \lambda u$$

for some $\lambda \in \mathbb{R}$, i.e. u is an eigenvector of M with eigenvalue λ.