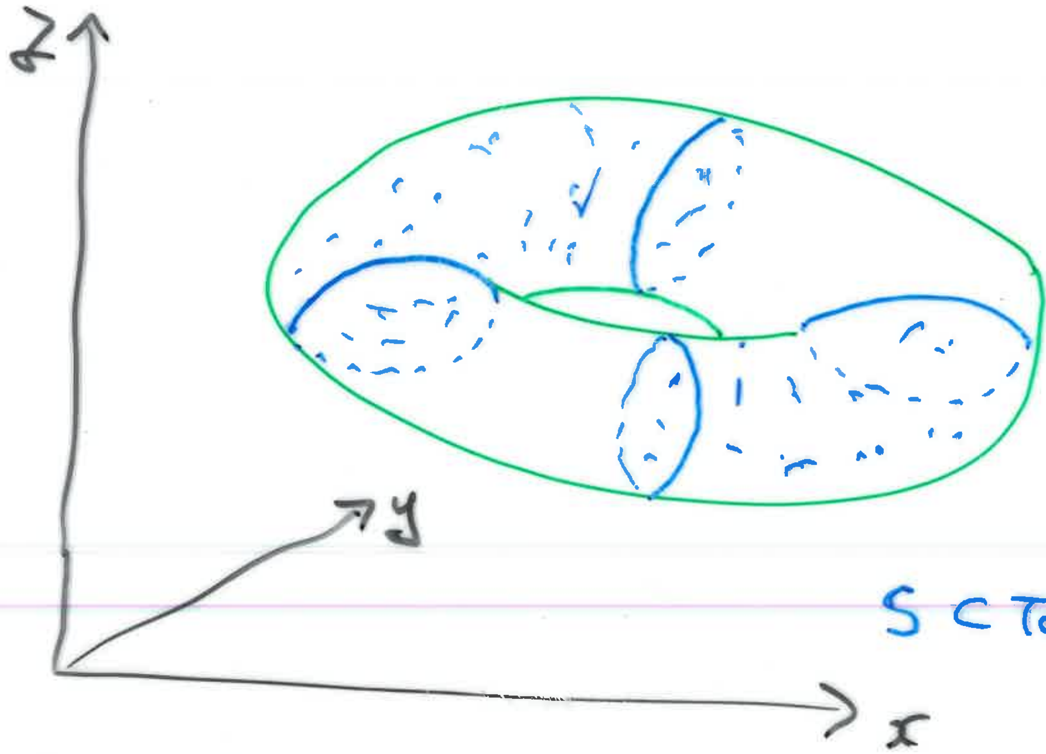
Example Consider a sample S of 750 points selected at random from two "quarter segments" of a torus.



$S \subset \text{Torus} \subset \mathbb{R}^3$

Note that any linear transformation $f: \mathbb{R}^3 \to \mathbb{R}^2$ will lose significant geometric information about S, since the torus can't be embedded in the plane.

From S let's compute a distance matrix $D = (d_{ij})$, Euclidean metric, and view the graph of K_ε (clique complex) for various $\varepsilon \geq 0$.

Homotopy

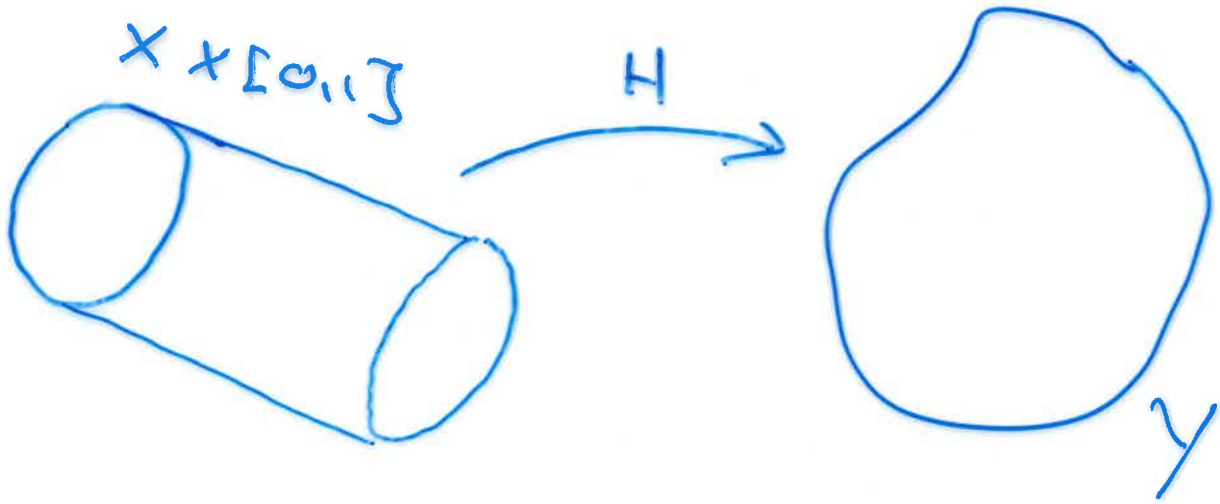
<u>Defn</u> Two maps $f: X \to Y$, and $g: X \to Y$ between topological spaces are <u>homotopic</u> if there exists a continuous map

$$H: X \times [0,1] \to Y, \quad (x,t) \mapsto H_t(x)$$

such that

$$H_0(x) = f(x) \quad \text{and}$$
$$H_1(x) = g(x)$$

for all $x \in X$.



We can think of $H_t(x)$ as a family of map

$$H_t(x) : X \to Y, \quad x \mapsto H_t(x).$$

We refer to H as the homotopy, and we write $f \simeq g$.

Example Let $Y \subseteq \mathbb{R}^n$ be a convex set. Let X be any topological space.

Any two maps $f: X \to Y$, $g: X \to Y$ are homotopic.

To see this, let $f, g : X \to Y$ be such maps.

We can define a homotopy

$$H : X \times [0,1] \longrightarrow Y,$$

$$(x, t) \longmapsto f(x) + t(g(x) - f(x))$$

$$= (1-t) f(x) + t g(x)$$

describes line from $f(x)$ to $g(x)$, and this in Y since Y is convex

So $H(x,t) \in Y$ since Y is convex.

Also

$$H(x,0) = f(x)$$

$$H(x,1) = g(x).$$

So $f \simeq g$ since H is continuous!

Defn Two topological spaces X, Y are **homotopy equivalent** if there exist maps

$$f : X \to Y, \quad g : Y \to X$$

such that

$$fg \simeq 1_Y \quad \text{and} \quad gf \simeq 1_X .$$

Here $1_Y : Y \to Y$ is the identity mapping on Y. And 1_X is the identity mapping on X.

Example $X = \mathbb{C} \setminus \{0\}$

$$Y = S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$$

Claim: X is homotopy equivalent to Y.

To verify the claim, we consider the following maps.

$g : S^1 = Y \longrightarrow X = \mathbb{C}\setminus\{0\}, \ z \longmapsto z$

$f : \mathbb{C}\setminus\{0\} = X \longrightarrow Y = S^1, \ z \longmapsto \frac{1}{|z|} z$

Clearly

$fg(z) = z, \quad fg = 1_Y$

$gf(z) = \frac{1}{|z|} z, \quad gf \simeq 1_X.$

To see that $gf \simeq 1_X$ we use the homotopy

$H : X \times [0,1] \longrightarrow X$

$(z, t) \longmapsto \left(\frac{1-t}{|z|} + t\right) z$

and note

$H_0(z) = \frac{z}{|z|} = gf(z)$

$H_1(z) = z = 1_X(z).$