First test: Monday 10 February 2020.
5 questions from homework sheet, taken from Sections 1–6.

Example The spaces

$(-1, 1)$ and $\mathbb{R}$

are homeomorphic because of the homeomorphism

$$f : (-1,1) \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{x}{1-x^2}.$$

Let's aim towards showing that the following four spaces are distinct (non-homeomorphic).

$[-1, 1]$

$\mathbb{R}$

$\mathbb{R}^2$

S^1

Recall A property is topological if, whenever X has the property, then so too does any Y homeomorphic to X.

Proposition Let $f : X \to Y$ be a homeomorphism. If X is connected then so too is Y.

Proof Let's suppose that Y is not connected. Then there exist open sets $U, V \subset Y$ such that

$$Y = U \cup V, \quad U \cap V = \phi, \quad U \neq \phi \neq V.$$

Suppose f is a homeomorphism. Since f is continuous

$$f^{-1}(U) = \{x \in X ; f(x) \in U\}$$

is open in X. So too is

$$f^{-1}(V).$$

Since f is a homeomorphism (and hence surjective)

$$f^{-1}(U) \cup f^{-1}(V) = X.$$

Also $f^{-1}(U) \cap f^{-1}(V) = \phi$.

Hence X is not connected. $\boxed{\checkmark}$

<u>Example</u> Let's show that $\mathbb{R}^2$ is <u>not</u> homeomorphic to $\mathbb{R}$.

Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ were a homeomorphism.

$X = \mathbb{R}^2 \setminus \{(1,1)\}$

$Y = \mathbb{R} \setminus \{f(1,1)\}$

<u>Exercise</u>: If $f: \mathbb{R}^2 \to \mathbb{R}$ is a homeomorphism then so too is $f: X \to Y$.

But X is connected and Y is <u>not</u> connected.

Hence there is <u>no</u> homeomorphism f from $\mathbb{R}^2$ to $\mathbb{R}$.

Towards Compactness

We'd like to say that $[-1,1]$ is "finite" and that $(-\infty,\infty)$ is "infinite". However, $(-1,1)$ is homeomorphic (bijective) with $(-\infty,\infty)$.

We'll use the word "compact" and "non-compact" instead.

Next lecture we'll define compactness and show that it's a topological property.

$[-1,1]$ $\mathbb{R}$

$(1,1)$