Class test: 12.00pm Monday 23 March

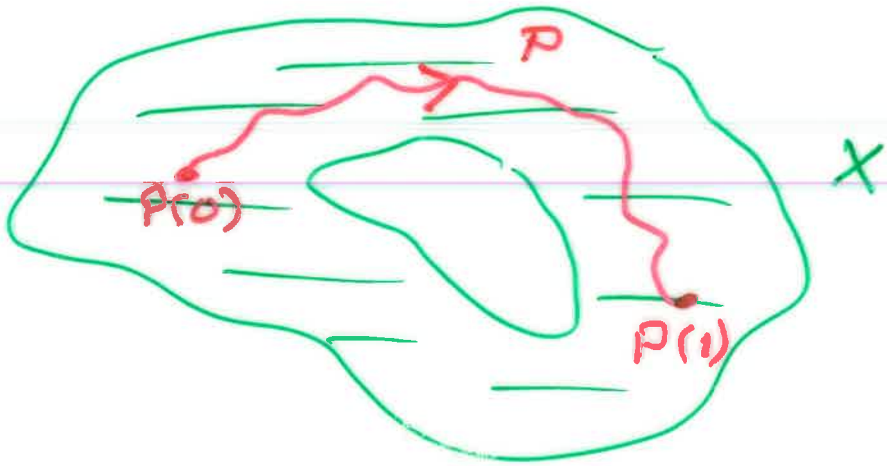
Topology and group theory are intimately related.

Let X be a topological space.

A continuous function

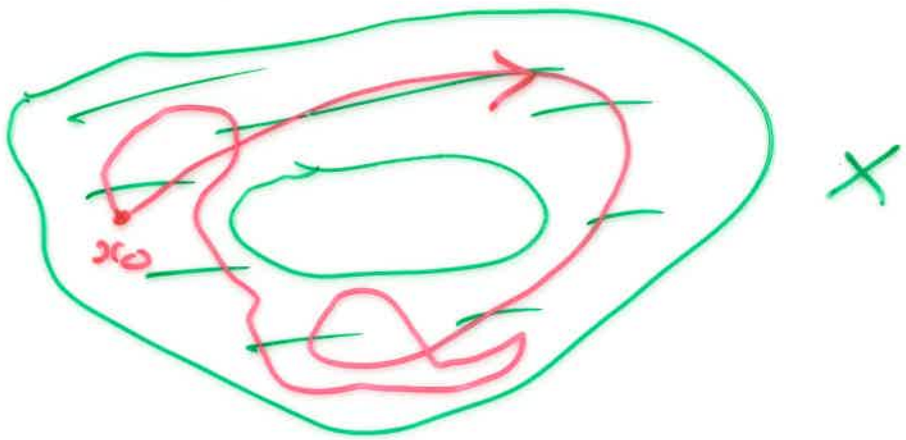
$$p : [0,1] \longrightarrow X$$

is called a path in X.



Choose $x_0 \in X$.

A path $p : [0,1] \longrightarrow X$ with $p(0) = p(1) = x_0$ is called a loop at x_0.



Given two loops

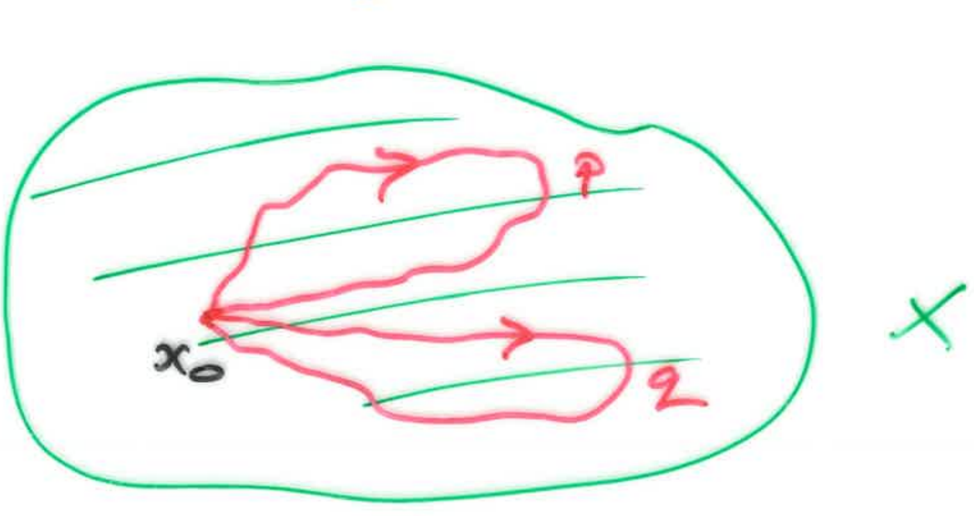
$$p, q : [0,1] \longrightarrow X$$

at x_0, we can combine them to form a new loop

$$p * q : [0,1] \longrightarrow X$$

at x_0 using the formula

$$p*q(s) = \begin{cases} p(2s), & 0 \le s \le \frac{1}{2} \\ q(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$



This multiplication does <u>not</u> satisfy the axioms of a group. It does not even have an identity element.

Two loops $p, q : [0,1] \to X$ at x_0 are <u>homotopic</u> <u>rel</u> <u>x_0</u> if there exists a continuous map

$$H : [0,1] \times [0,1] \to X, \ (s,t) \mapsto H_t(s)$$

with

$$H_0(s) = p(s),$$
$$H_1(s) = q(s),$$
$$H_t(0) = H_t(1) = x_0 \text{ for all } t \in [0,1].$$

Homotopy rel x_0 is an equivalence relation on loops at x_0.

Let $[p]$ denote the equivalence class of loop p.

Let

$$\pi_1(X, x_0) = \left\{ [p] \;:\; \begin{array}{l} p : [0,1] \to X \text{ is} \\ \text{a loop at } x_0 \end{array} \right\}$$

Theorem (Henri Poincaré)

$\pi_1(X, x_0)$ is a group under the multiplication

$$[p] * [q] = [p * q].$$

Proof Not difficult. See Armstrong's book.

Terminology

We call $\pi_1(X, x_0)$ the **fundamental group** of X based at x_0.

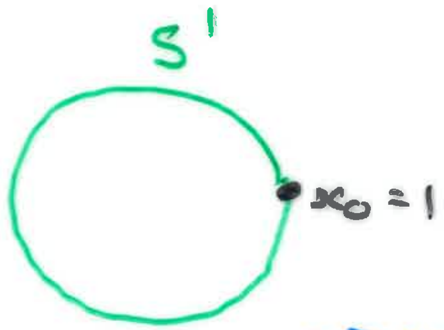
Example $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$

$$1 \in S^1$$

$\pi_1(S^1, 1) \cong \mathbb{Z}$ (additive group of integers, $(\mathbb{Z}, +)$)

See Armstrong for full details.

Idea:



$X = S^1$, $x_0 = 1$

$p_1 : [0,1] \longrightarrow S^1$, $\theta \longmapsto p_1(\theta) = e^{2\pi i \theta}$

Then

$p_2 = p_1 * p_1 : [0,1] \longrightarrow S^1$, $\theta \longmapsto e^{4\pi i \theta}$

And

$(p_1 * p_1) * p_1$ is homotopic rel 1

to

$p_3 : [0,1] \longrightarrow S^1$, $\theta \longmapsto e^{6\pi i \theta}$

In general we have a loop

$P_n : [0,1] \to S^1, \theta \mapsto e^{2n\pi i\theta}$

for each n.

One needs to show:

1) $[P_n] \neq [P_m]$ if $m \neq n$

2) Any loop

$q : [0,1] \to S^1$

based at 1 is homotopic

rel 1 to some P_n.

We say that q has

<u>winding number</u> n.

3) $[P_n * P_m] = [P_{n+m}]$.