So far:

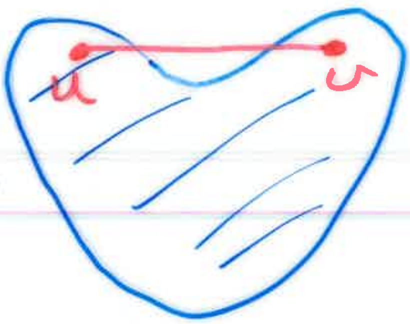
- topology has modern applications
- topology involves subtle maths

Next aims:

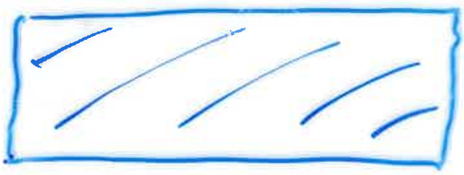
- precise definition of the Euler characteristic $\chi(X)$ a topological space X
- flavour of the ingredients of the proof that $\chi(X)$ is a topological invariant.
- Application of the Euler characteristic in Economics

Defn A set $X \subseteq \mathbb{R}^n$ is said to be convex if, for any $u, v \in X$, the straight line from u to v lies entirely in X.

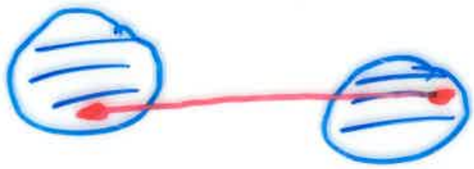
Example $(n=2)$



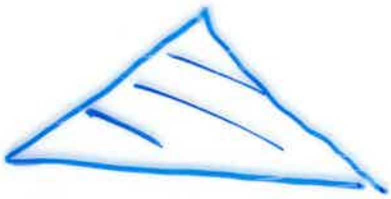
$X \subseteq \mathbb{R}^2$ not convex



$X \subseteq \mathbb{R}^2$ Convex



$X \subseteq \mathbb{R}^2$ not convex



$X \subseteq \mathbb{R}^2$ Convex

Suppose $v_0, v_1, \dots, v_k \in \mathbb{R}^n$

Let

$C = \mathrm{Conv}(v_0, v_1, \dots, v_k)$

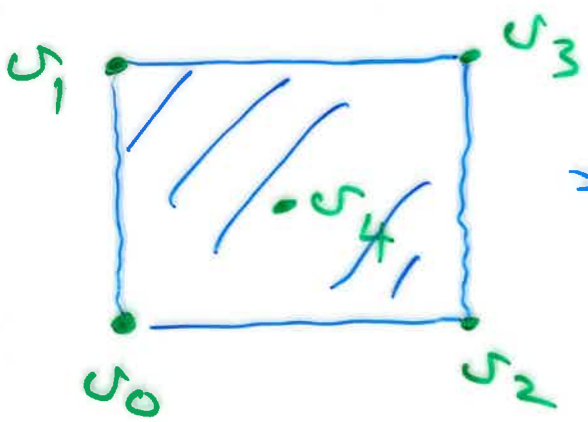
denote the "smallest" convex set in $\mathbb{R}^n$ containing $v_0, v_1, \dots, v_k$.

We call C the <u>convex hull</u> of $v_0, v_1, \dots, v_k$.

<u>Example</u> $v_0 = (0,0)$, $v_1 = (0,1)$,

$v_2 = (1,0)$, $v_3 = (1,1)$,

$v_4 = (\frac{1}{2}, \frac{1}{2}) \in \mathbb{R}^2$

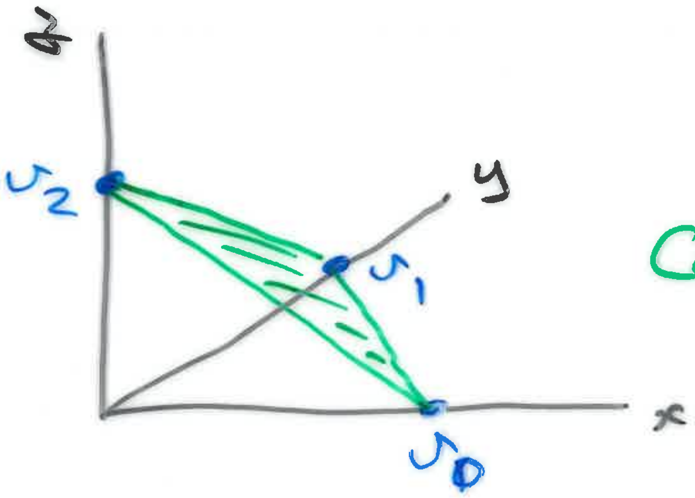


$= \mathrm{Conv}(v_0, v_1, \dots, v_4)$

Example $v_0 = (1,0,0)$

$v_1 = (0,1,0)$ $\in \mathbb{R}^3$

$v_2 = (0,0,1)$



$\mathrm{Conv}(v_0, v_1, v_2)$

In general

$C = \mathrm{Conv}(v_0, v_1, \dots, v_k)$

consists of all those points in $\mathbb{R}^n$

$$C = \left\{ \lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_k v_k \;:\; \begin{array}{l} \lambda_i \in \mathbb{R} \\ \lambda_i \geq 0 \\ \sum_{i=0}^{k} \lambda_i = 1 \end{array} \right\}$$

<u>Defn</u> Let $v_0, v_1, \dots, v_k \in \mathbb{R}$

be linearly independent vectors.

We call

$$C = \mathrm{Conv}(v_0, v_1, \dots, v_k)$$

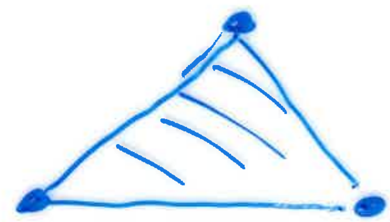
a simplex of dimension k,

or k-simplex.

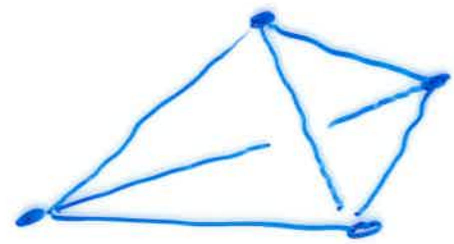
0-simplex = point

1-simplex = line segment

2-simplex = Solid triangle



3-simplex = Solid tetrahedron



simplices have "faces"!

If A and B are simplices, and if the vertices of A form a subset of the vertices of B, then we say that A is a face of B.

Example A 3-simplex has

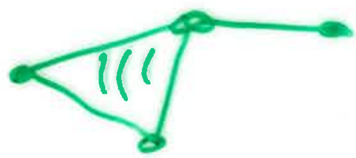
- 4 faces of dimension 2
- 6 faces of dimension 1
- 4 faces of dimension 0
- 1 faces of dimension 3

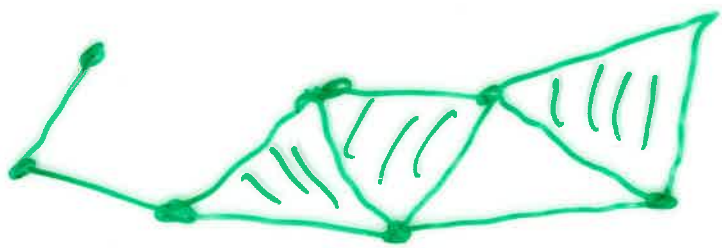
Defn A finite collection of simplices in $\mathbb{R}^n$ is called a simplicial complex if

1) whenever a simplex lies in the collection, then so too do all its faces.

2) whenever two simplices in the collection intersect, they do so in a common face.

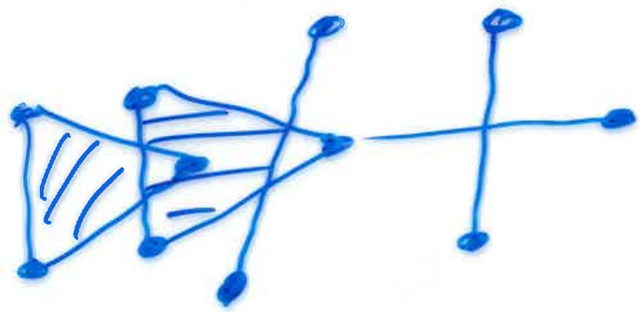
Example





Simplicial complex

Non-example



Any simplicial complex is a subset of $\mathbb{R}^n$, and so it is a topological space with the subspace topology.

We let $K, L, \dots$ denote simplicial complexes.

We let $|K|, |L|, \dots$ to denote the corresponding topological spaces.

Defn Let X be a topological space. A triangulation of X consists of a simplicial complex K and a homeomorphism

$$h : |K| \longrightarrow X$$

We define the Euler characteristic of X to be

$$\chi(X) = \chi(K)$$

$$= \alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 + \cdots$$

where

α_k = # of k-simplices in K.