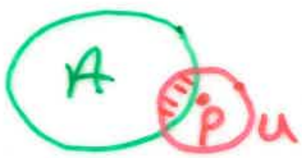
Recall

$A \subseteq X$ is closed iff $X \setminus A$ is open.

$p \in X$ is an accumulation point for $A \subseteq X$ if for any open $p \in U \subseteq X$ we have $U \setminus \{p\} \cap A \neq \emptyset$.



Proposition A set A in a topological space X is closed if, and only if, A contains all its accumulation points.

proof Suppose A is closed. Then $X \setminus A$ is open. Any point $x \in X \setminus A$ lies in the open set $X \setminus A$, so no point $x \in X \setminus A$ can be an accumulation point. So any accumulation point must lie in A.

Conversely, suppose A contains all its accumulation points.

Let $x \in X \setminus A$. Since x is not an accumulation point we can find an open set

$x \in U_x \subseteq X \setminus A$

So

$$X \setminus A = \bigcup_{x \in X \setminus A} U_x .$$

So $X \setminus A$ is open.

Hence A is closed. $\boxed{\text{QED}}$

<u>Aim</u> Show that Peano's function

$$f : [0,1] \longrightarrow \Delta$$

is surjective.

$$f = \lim_{n \to \infty} f_n .$$

It should be clear that Δ equals the union of image(f) together with the accumulation points of image(f).

So we just need to show that image(f) contains all its accumulation points.

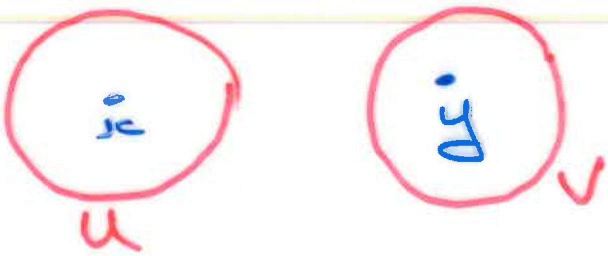
i.e. we just need to show that image(f) is closed.

We know that $[0,1]$ is compact, and hence (by one of our propositions), that $\underbrace{f([0,1])}_{\text{image}(f)}$ is compact.

$$f : [0,1] \twoheadrightarrow \text{image}(f) \subseteq \Delta$$

So we just need to show that "compact" implies "closed" under a suitable hypothesis.

Defn A topological space X is said to be **Hausdorff** if for any distinct $x, y \in X$ there exist open sets U, V in X with $x \in U$, $y \in V$ and $U \cap V = \emptyset$.



Example $\mathbb{R}$ with standard topology is Hausdorff. So too is $\mathbb{R}^n$.

Example $\mathbb{Z}$ with cofinite topology (i.e. $U \subseteq \mathbb{Z}$ is open iff $\mathbb{Z} \setminus U$ is finite or $U = \emptyset$) is not Hausdorff.

Exercise if X is Hausdorff and if Y is homeomorphic to X then Y is Hausdorff.

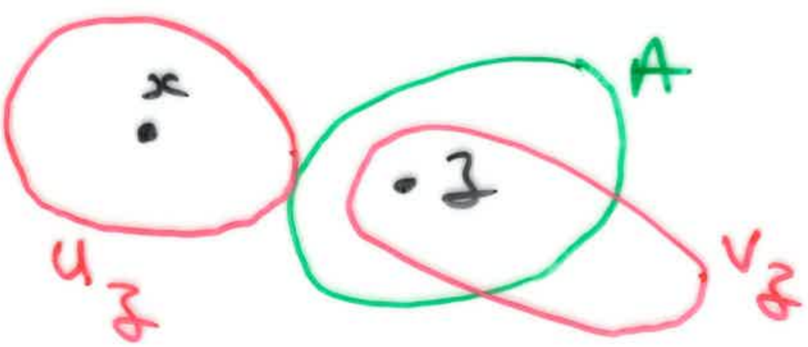
Proposition A compact subset of a Hausdorff topological space is closed.

Proof Let X be a Hausdorff space. Let A be a compact subset of X. We need to

Prove that A contains all its accumulation points.

Let
$$x \in X \setminus A.$$
Just need to show that x is not an accumulation point of A.

Let $z \in A$. Since X is Hausdorff we can find open set U_z, V_z such that $x \in U_z$, $z \in V_z$ and
$$U_z \cap V_z = \phi.$$



We have a collection of open sets

$$\{V_z\}_{z \in A}$$

But A is compact, so (!) we can find a finite collection of points

$$z_1, z_2, \ldots, z_k \in A$$

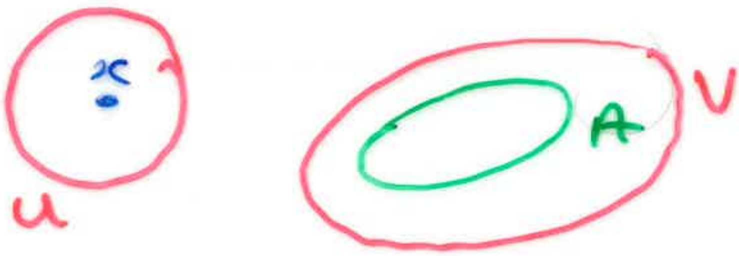
with

$$A \subseteq V_{z_1} \cup V_{z_2} \cup \cdots \cup V_{z_k} =: V$$

Now V is disjoint from the following finite intersection:

$$U := U_{z_1} \cap U_{z_2} \cap \cdots \cap U_{z_k}.$$

But U is a finite intersection of open sets and is thus open.



Since $U \cap V = \emptyset$ we see that x is <u>not</u> an accumulation point of A. Hence A contains all its accumulation points. Hence A is closed.

$\blacksquare$