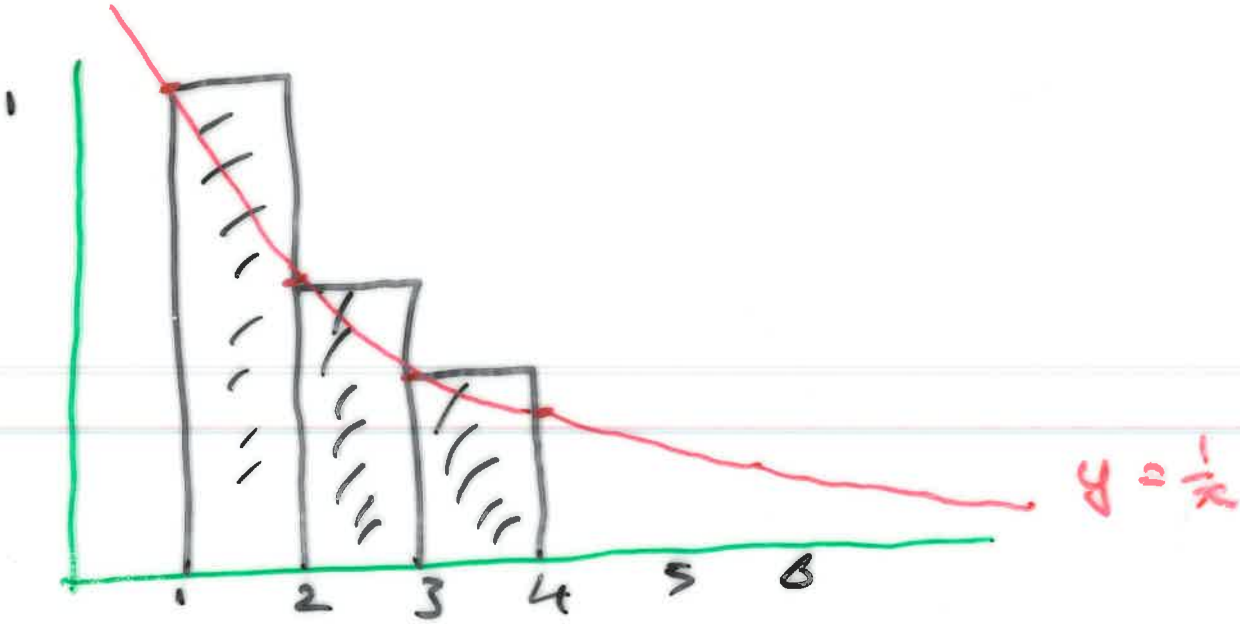
Aside: Consider

$a_1 = 1, \quad a_2 = 1\frac{1}{2}, \quad a_3 = 1\frac{5}{6}, \quad \ldots, \quad a_n = a_{n-1} + \frac{1}{n}$

So $\lim_{n\to\infty} a_n - a_{n+1} = \lim_{n\to\infty} \frac{1}{n+1} = 0$

$\lim_{n\to\infty} a_n$ does not exist



a_n = area of first n boxes.

$$a_n > \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$$

$$\lim_{n\to\infty} a_n > \lim_{n\to\infty} \ln(n+1) = \infty$$

<u>Defn</u> A sequence of points $a_1, a_2, \ldots\ldots$ in $\mathbb{E}^d$ is said to be a Cauchy sequence if, for any $\varepsilon > 0$, there is an integer N such that

$$\| a_m - a_n \| < \varepsilon$$

for all $m, n > N$.

<u>Theorem</u> Any Cauchy sequence $a_1, a_2, \ldots$ in $\mathbb{E}^d$ has a limit $\lim_{n \to \infty} a_n$.

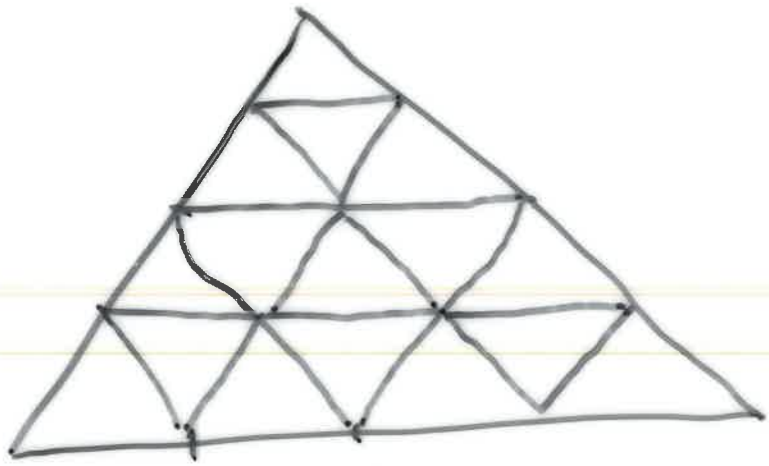
Last time we were constructing

$$f : [0,1] \longrightarrow \Delta$$

as

$$f(t) = \lim_{n\to\infty} f_n(t) .$$

where $f_n(t)$ was defined by subdividing Δ into small subtriangles



of side $\frac{1}{2^{n-1}}$. For fixed $t \in [0,1]$ the sequence $f_1(t), f_2(t), f_3(t), \dots$ is a Cauchy sequence and hence converges by the above theorem.

If t is close to t' then $f(t)$ is close to $f(t')$ and thus, intuitively, the function is continuous. It's not difficult to convert this to a rigorous argument.

It remains to prove that f is surjective. To show this we'll use compactness of $[0,1]$ and "related results".

<u>Some</u> <u>Theory</u>

<u>Defn</u> Let X be a topological space. A subset $A \subseteq X$ is said to be <u>closed</u> if its complement $X \setminus A$ is open.

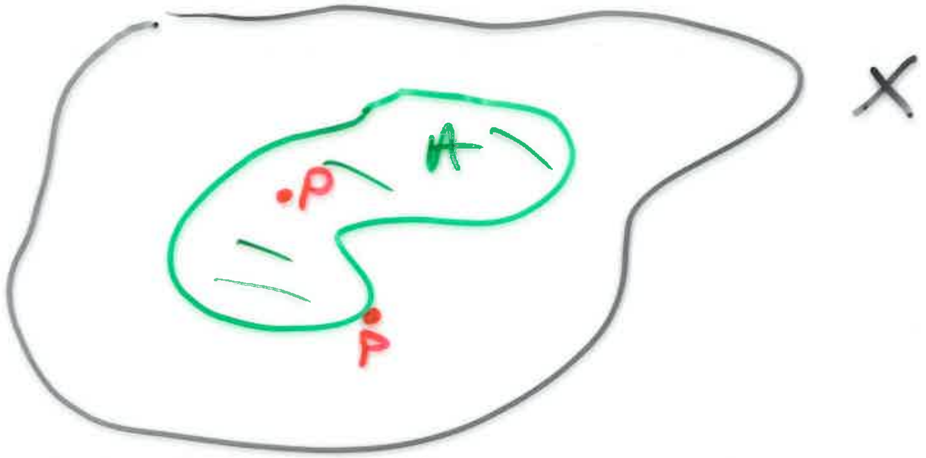
Example The subset $A = [0,1] \subseteq \mathbb{R}$ is a closed subset with respect to the usual topology on $\mathbb{R}$. This is because

$$\mathbb{R} \setminus [0,1] = (-\infty, 0) \cup (1, \infty)$$

is open in $\mathbb{R}$.

Example $(0,1]$ is neither open nor closed.

Defn Let A be a subset of a topological space X. A point $p \in X$ is an accumulation point of A if every open subset U of X containing p also contains some point in $A \setminus \{p\}$



Example Let $X = \mathbb{R}$, and $A = [0,1]$. Then every point of A is an accumulation point of A. So too is 0.

Example Let $X = \mathbb{R}$. Consider

$$A = \left\{ \frac{1}{n} \right\}_{n=1,2,3,\ldots}$$

In this example 0 is the only accumulation point

Proposition A set A in a topological space X is closed if, and only if, it contains all of its accumulation points.