Third in-class test: Wed 27 Nov

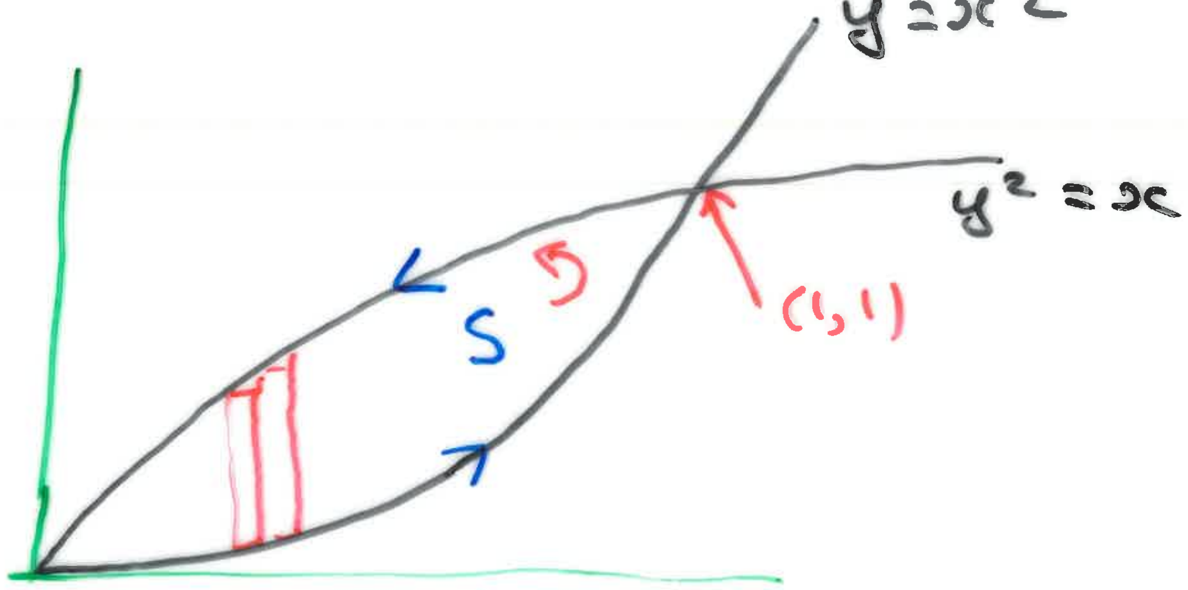
Problem 10.6

Verify Green's Theorem in the plane for

$$w = (2xy - x^2)dx + (x+y^2)dy$$

and S the region bounded by $y = x^2$, $y^2 = x$.

Soln



Need to verify

$$\int_{\partial S} w = \int_S dw$$

$$\text{LHS} = \int_{\partial S} \omega$$

$$= \int_{\partial S} (2xy - x^2)\,dx + (x + y^2)\,dy$$

$y = x^2$, $x = t$, $y = t^2$, $dx = dt$,
$dy = 2t\,dt$

$$= \int_0^1 \left(2t^3 - t^2 + 2(t + t^4)\cancel{dt}\right) dt$$

$y^2 = x$, $y = t$, $x = t^2$, $dy = dt$,
$dx = 2t\,dt$

$$+ \int_1^0 2(2t^3 - \cancel{t}^4) + (t^2 + t^2) \quad dt$$

$$= \dots$$

$$= \frac{1}{30}.$$

$$RHS = \int_S d\omega$$

$$= \int_S d\left((2xy - x^2)\, dx + (x + y^2)\, dy \right)$$

$$= \int_S 2x\, dy \wedge dx + dx \wedge dy$$

$$= \int_S (1 - 2x)\, dx \wedge dy$$

$$= \int_0^1 \left(\int_{y=x^2}^{y=\sqrt{x}} (1 - 2x) \quad dy \right) dx$$

$$= \int_0^1 y - 2xy \Big|_{y=x^2}^{y=\sqrt{x}} \quad dx$$

$$= \int_0^1 \sqrt{x} - 2x^{\frac{3}{2}} - x^2 + 2x^3 \, dx$$

$$= \ldots$$

$$= \frac{1}{30}$$

Problem Show that the area of the region S bounded by a simple closed curve C in the xy-plane

is $\frac{1}{2}\int_C x\,dy - y\,dx$.

$\underline{\underline{\text{Sol}^n}}$

$$\frac{1}{2}\int_C x\,dy - y\,dx$$

$$\underset{\text{Stokes' formula}}{=} \frac{1}{2}\int_S d(x\,dy - y\,dx)$$

$$= \frac{1}{2}\int_S dx\wedge dy - dy\wedge dx$$

$$= \int_S dx\wedge dy$$

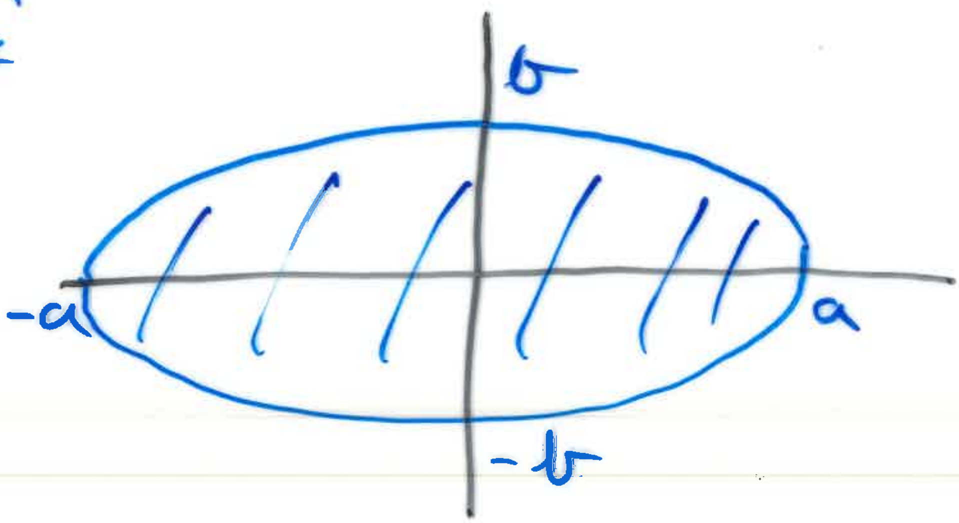
$$= \text{area of } S.$$

QED

Example Find the area of the region S bounded by the ellipse

$$x = a\cos t, \quad y = b\sin t$$

Sol^n



From the preceding example

required area =

$$\frac{1}{2}\int_C x\,dy - y\,dx$$

$$= \frac{1}{2} \int_0^{2\pi} a \cos(t)\, b \cos(t)\, dt + ab \sin(t) \sin(t)\, dt$$

$$= \frac{ab}{2} \int_0^{2\pi} \cos^2(t) + \sin^2(t)\, dt$$

$$= \frac{ab}{2} \int_0^{2\pi} dt$$

$$= \frac{ab}{2} \cdot 2\pi$$

$$= ab\pi$$