Third in-class test : Wed 27 Nov

Divergence

Given a vector field

$F = F_1 \underline{i} + F_2 \underline{j} + F_3 \underline{k}$

on $\mathbb{R}^3$ we define the associated

flux 2-form

$\omega = F_3 \, dx \wedge dy + F_1 \, dy \wedge dz + F_2 \, dz \wedge dx$

The derivative of ω is

a 3-form

$d\omega = A \, dx \wedge dy \wedge dz$

Definition We define the divergence of F to be the function

$$\mathrm{div}(F) = A$$

Example Consider

$$F = xz\,\underline{\underline{i}} - y^2\,\underline{\underline{j}} + 2x^2y\,\underline{\underline{k}}.$$

Let's find $\operatorname{div}(F)$.

Sol^n

$$\omega = 2x^2y\,dx\wedge dy + xz\,dy\wedge dz - y^2\,dz\wedge dx$$

$$d\omega = 4xy\,dx\wedge \underbrace{\cancel{dx\wedge dy}}_{=0} + 2x^2\,dy\wedge \underbrace{\cancel{dx\wedge dy}}_{=0}$$

$$+ z\,dx\wedge dy\wedge dz + x\,dz\wedge \underbrace{\cancel{dy\wedge dz}}_{=0}$$

$$+ -2y\,dy\wedge dz\wedge dx$$

$$= (z-2y)\,dx\wedge dy\wedge dz$$

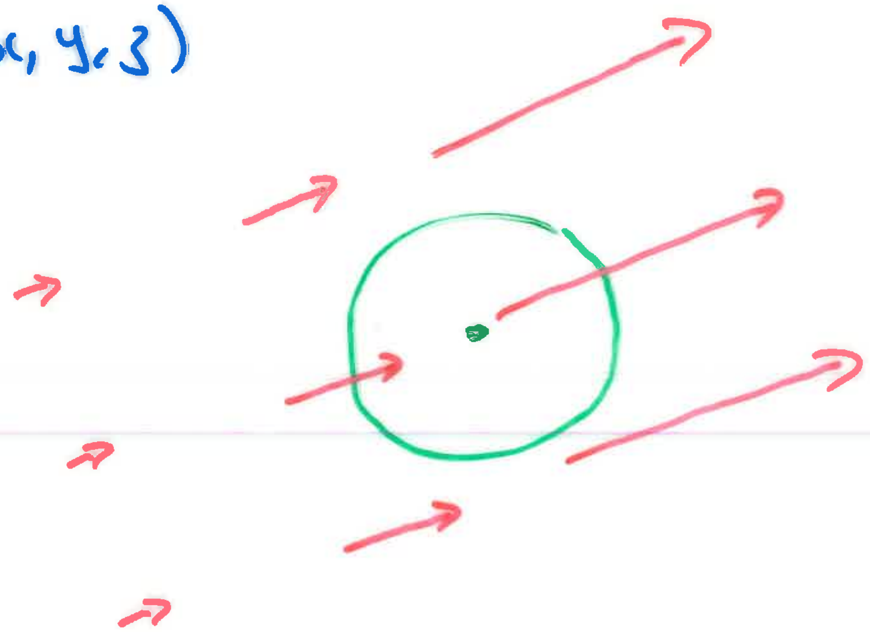
Thus

$$\boxed{\operatorname{div}(F) = z - 2y}$$

Interpretation of div(F)

Let F represent the flow of a fluid in $\mathbb{R}^3$.

Place a small ball with centre fixed at the point (x, y, z)



Fluid flows into the ball and out of the ball. The difference is measured by the number

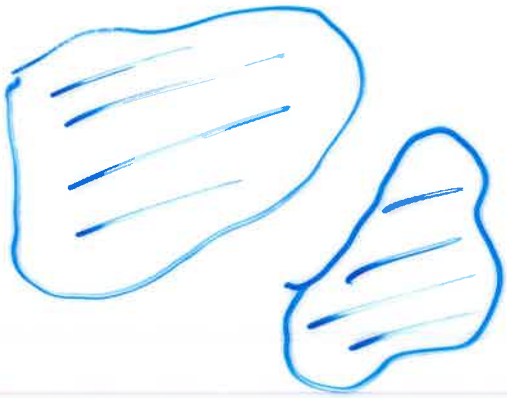
$$\operatorname{div}(F)(x, y, z)$$

Regions in plane

1)

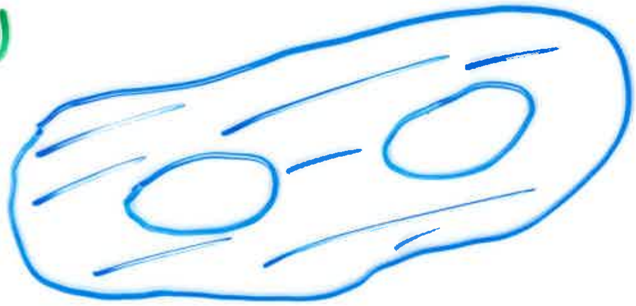
- not connected
- not simply connected

2)



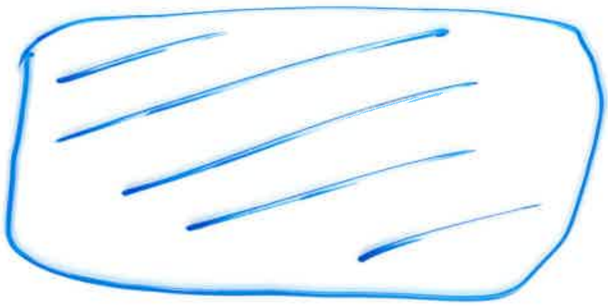
- not connected
- simply connected

3)



- connected
- not simply connected

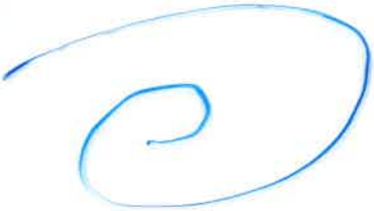
4)



- connected
- simply connected

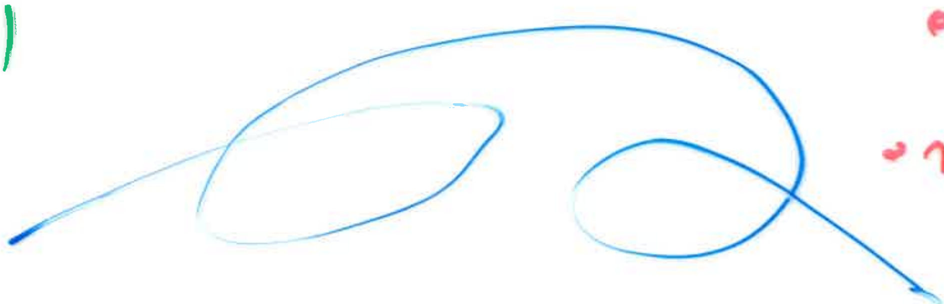
Curves

1)



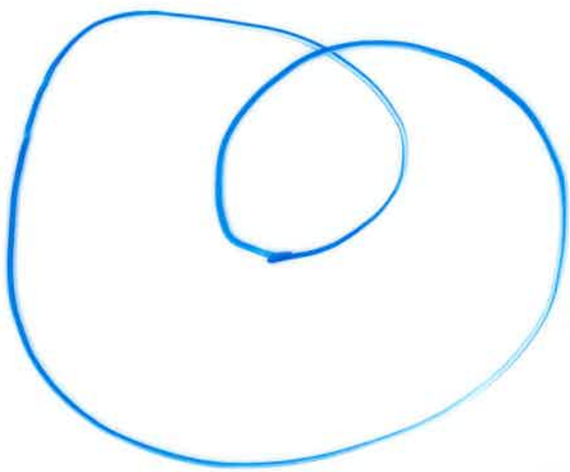
- not closed
- simple curve

2)



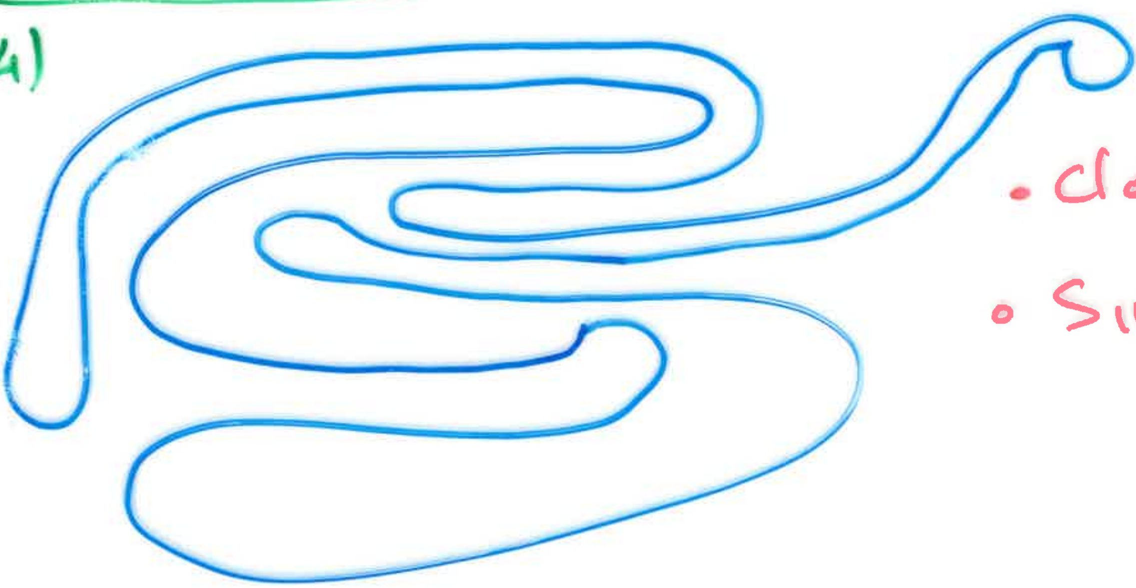
- not closed
- not a simple curve

3)



- closed
- not a simple curve

4)



- closed
- simple

Green's Theorem in the plane

Let $P = P(x,y)$, $Q = Q(x,y)$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$ be single valued and continuous in a simply connected region S bounded by a simple closed curve C. Then

$$(*) \quad \oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

i.e. for $\omega = P\,dx + Q\,dy$

$$\int_{\partial S} \omega = \int_S d\omega$$

One page 251 you'll find Stoke's Theorem, which is ~~a~~ Green's Theorem to the case where S is a simply connected region on a surface. Equation (*) becomes a bit more involved.

Alternatively, (*) can be written

$$\int_{\partial S} \omega = \int_{S} d\omega$$

Divergence Theorem

Let F be a vector field that is continuously differentiably on a closed region V in 3-dimensional space bounded by a smooth surface S.

Then

$$\iiint_V \nabla \cdot F \, dV = \iint_S F \cdot \underline{n} \, dS$$

where $\underline{n}$ is an outward pointing normal to S. Maybe some extra hypotheses on V are needed.

i.e. for the flux 2-form ω

$$\int_V d\omega = \int_{\partial V} \omega$$