Test: Wednesday 14th November.

Curl

Given a vector field

$F = F_1 \underline{\underline{i}} + F_2 \underline{\underline{j}} + F_3 \underline{\underline{k}}$

on $\mathbb{R}^3$ (where

$F_i = F_i(x, y, z)$ is a real-valued function

$\underline{i} = (1,0,0)$, $\underline{j} = (0,1,0)$, $k = (0,0,1)$).

We can define the associated 1-form

$$\omega = F_1 dx + F_2 dy + F_3 dz .$$

Suppose that the total derivative of ω is

$$d\omega = A\, dx \wedge dy + B\, dy \wedge dz + C\, dz \wedge dx .$$

Then we define the vector field

$$\mathrm{curl}(F) = B \underline{\underline{i}} + C \underline{\underline{j}} + A \underline{\underline{k}}$$

Example $F = xy\,\underline{i} + y^2\,\underline{j} + 2x^2y\,\underline{k}$

Find curl(F).

Sol^n

$$\omega = xy\,dx + y^2\,dy + 2x^2y\,dz$$

$$d\omega = (y\,dx + x\,dy)\wedge dx + (2y\,dy)\wedge dy + (4xy\,dx + 2x^2\,dy)\wedge dz$$

$$= \underbrace{\cancel{y\,dx\wedge dx}}_{=0} + x\,dy\wedge dx + \underbrace{\cancel{2y\,dy\wedge dy}}_{=0} + 4xy\,dx\wedge dz + 2x^2\,dy\wedge dz$$

$$= -x\,dx\wedge dy + 2x^2\,dy\wedge dz - 4xy\,dz\wedge dx$$

$$\mathrm{Curl}(F) = 2x^2\,\underline{i} - 4xy\,\underline{j} - x\,\underline{k}$$

$$\mathrm{Curl}(F) = (2x^2, -4xy, -x).$$

Interpretation of Curl

Imagine a vector field F on $\mathbb{R}^3$ describes the directions and speeds of particles in a fluid.

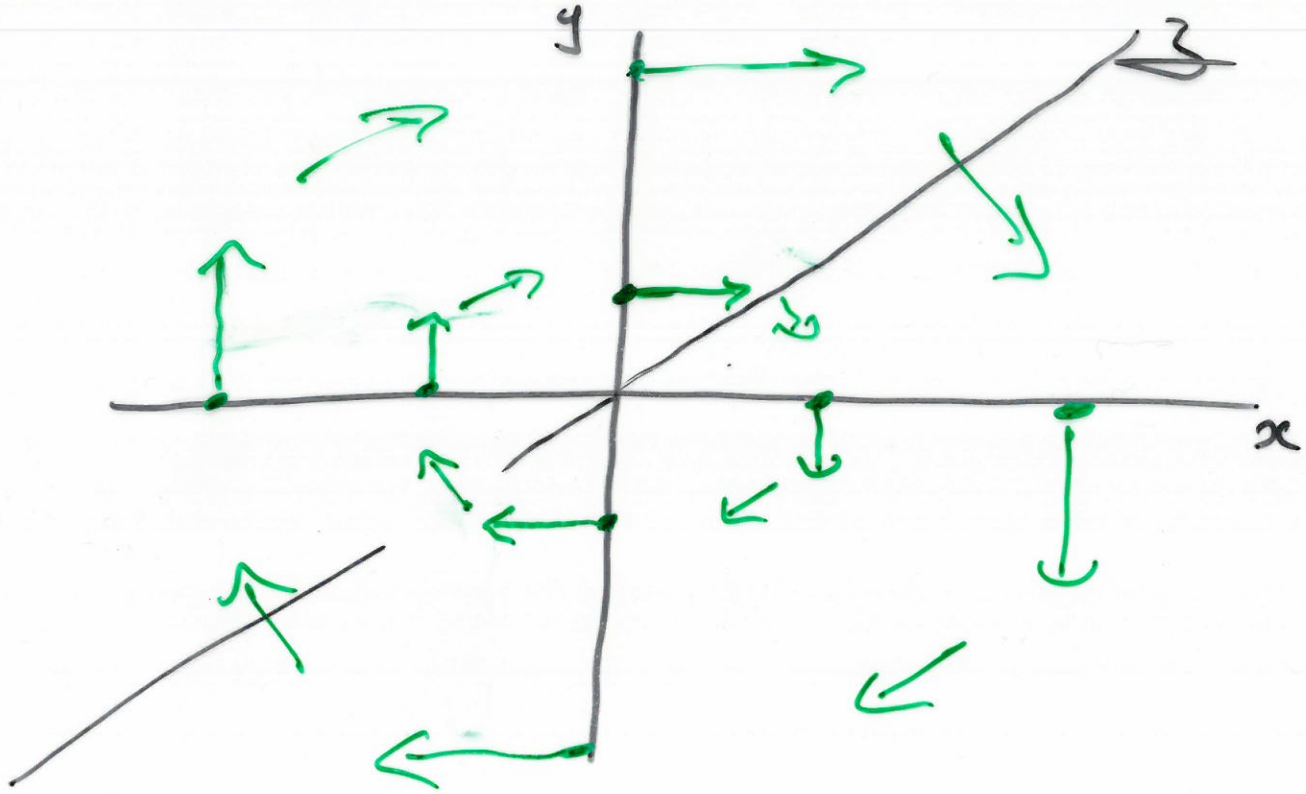
Place a rough small spherical ball in the fluid, with the centre of the ball fixed at point (x, y, z). The ball can rotate about its centre.

The fluid will cause the ball (with fixed centre) to rotate.

The axis of the rotation is the direction of the vector $\mathrm{Curl}(F)$. The angular speed of rotation is given by the size of the vector $\mathrm{Curl}(F)$.

Example Consider

$$F(x,y,z) = y\,\underline{\underline{i}} - x\,\underline{\underline{j}}$$



$$\omega = y\,dx - x\,dy$$

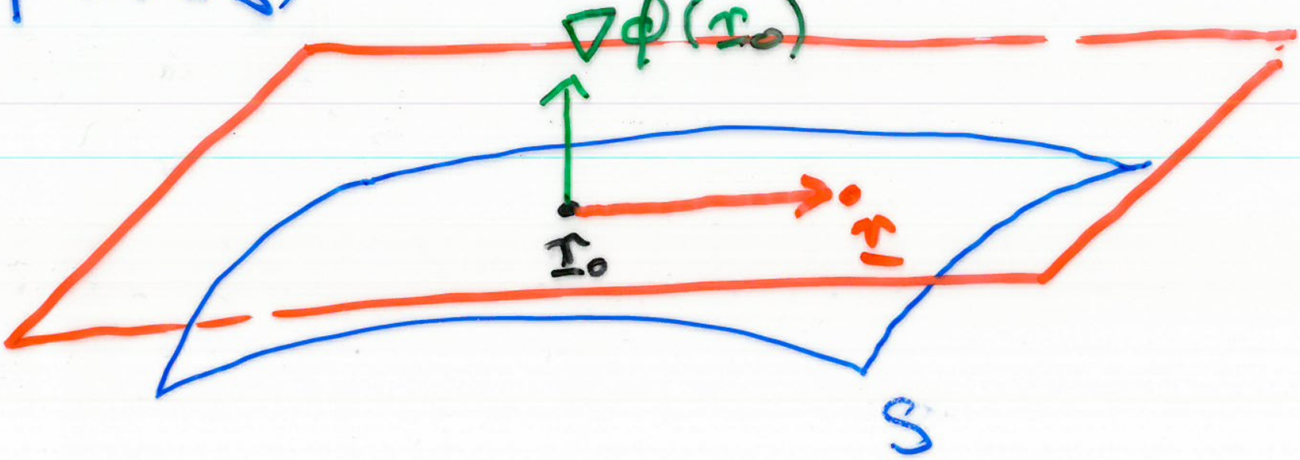
$$d\omega = dy \wedge dx - dx \wedge dy$$

$$= -2\,dx \wedge dy$$

$$\mathrm{Curl}(F) = -2\,\underline{\underline{k}}$$

Gradient again

$\phi(x, y, z) = k$



The tangent plane to S at some point $\underline{r}_0 = (x_0, y_0, z_0) \in S$, where S is described by

$$\phi(x, y, z) = k$$

consists of all points $\underline{r} = (x, y, z)$ such that

$$\boxed{\nabla\phi(\underline{r}_0) \cdot (\underline{r} - \underline{r}_0) = 0}$$

Example find a ~~unit~~ normal vector to the surface S :

$$2x^2 + 4yz - 5z^2 = -10$$

at the point $\underline{r}_0 = (3, -1, 2)$.

Then find the equation of the tangent plane to S at $\underline{r}_0 = (3, -1, 2)$.

Sol^n

Let $\phi(x, y, z) = 2x^2 + 4yz - 5z^2$

$$\text{grad}(\phi) = \nabla\phi = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}$$

$$= 4x\,\underline{i} + 4z\,\underline{j} + (4y - 10z)\,\underline{k}$$

$$\nabla\phi(\underline{r}_0) = 12\,\underline{i} + 8\,\underline{j} - 24\,\underline{k}$$

$$= (12, 8, -24).$$

Tangent plane at $\underline{r}_0$ is

$$(12, 8, -24) \cdot \big((x, y, z) - (3, -1, 2)\big) = 0$$

$$(3, 2, -6) \cdot (x-3, y+1, z-2) = 0$$

$$3(x-3) + 2(y+1) - 6(z-2) = 0$$

$$3x - 9 + 2y + 2 - 6z + 12 = 0$$

$$\boxed{3x + 2y - 6z = -5}$$

$$\nabla\phi(\underline{r}_0) = (12, 8, -24) = 4(3, 2, -6)$$

Length $|\nabla\phi(r_0)| = 4\sqrt{9 + 4 + 36}$

$$= 4 \cdot 7 = 28$$

$$\boxed{\text{Unit normal is } \frac{1}{28}(12, 8, -24)}$$