Third test: Wed 27 November

- We know how to calculate

  $d\omega$

  for any n-form ω, in k variables.
- We know where the rules for differentiation come from, namely Stokes's formula.
- But how should we interpret $d\omega$?

Next aim: stick to $k=3$ variables, and interpret $d\omega$ for $n=0,1,2$.

Answer: Div, grad, curl and all that.

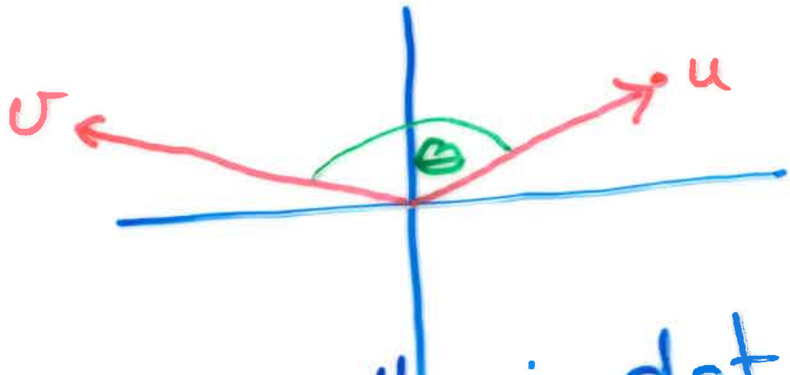
Dot products of vectors

Given two vectors

$u = (u_1, u_2) \in \mathbb{R}^2$

$v = (v_1, v_2) \in \mathbb{R}^2$

in the plane $\mathbb{R}^2$.



we define their dot product to be the number

$$u \cdot v = u_1 v_1 + u_2 v_2$$

Example If $u = (2, 3)$, $v = (4, 5)$

then $u \cdot v = 2.4 + 3.5 = 23$

We define the length of u to be

$$|u| = \sqrt{u_1^2 + u_2^2} = \sqrt{u \cdot u}$$

It is easy to prove:

Theorem $u \cdot v = |u|\,|v| \cos\theta$

In particular, u and v are perpendicular to each other if and only if $u \cdot v = 0$

For two vectors

$u = (u_1, u_2, u_3) \in \mathbb{R}^3$

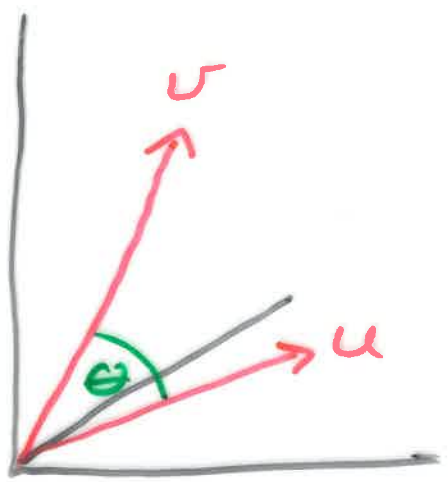
$v = (v_1, v_2, v_3) \in \mathbb{R}^3$

we define the dot product

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

and we define

$$|u| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{u \cdot u}$$



Theorem

$u \cdot v = |u|\,|v| \cos\theta$

So u, v are at right angles if and only if $u \cdot v = 0$.

Gradient

Let $\phi(x, y, z)$ be a real valued differentiable function

$$\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}.$$

The gradient of ϕ is defined as

$$\text{grad } \phi = \nabla \phi = \frac{\partial \phi}{\partial x} \underline{\underline{i}} + \frac{\partial \phi}{\partial y} \underline{\underline{j}} + \frac{\partial \phi}{\partial z} \underline{\underline{k}}$$

where $\underline{\underline{i}} = (1, 0, 0)$

$\underline{\underline{j}} = (0, 1, 0)$

$\underline{\underline{k}} = (0, 0, 1)$

Example $\phi(x, y, z) = x^2 + y^2 + z^2$

$$\text{grad } \phi = \nabla \phi = 2x \underline{\underline{i}} + 2y \underline{\underline{j}} + 2z \underline{\underline{k}}$$

$$= (2x, 2y, 2z)$$

We can think of the grad ϕ as the derivative of a 0-form ϕ

$$\nabla \phi = d\phi$$

Interpretation of the gradient

Consider a surface S defined by an equation

$$\phi(x, y, z) = k \quad (k \text{ a constant})$$

Example Let $\phi(x,y,z) = x^2+y^2+z^2$

Let $k=9$

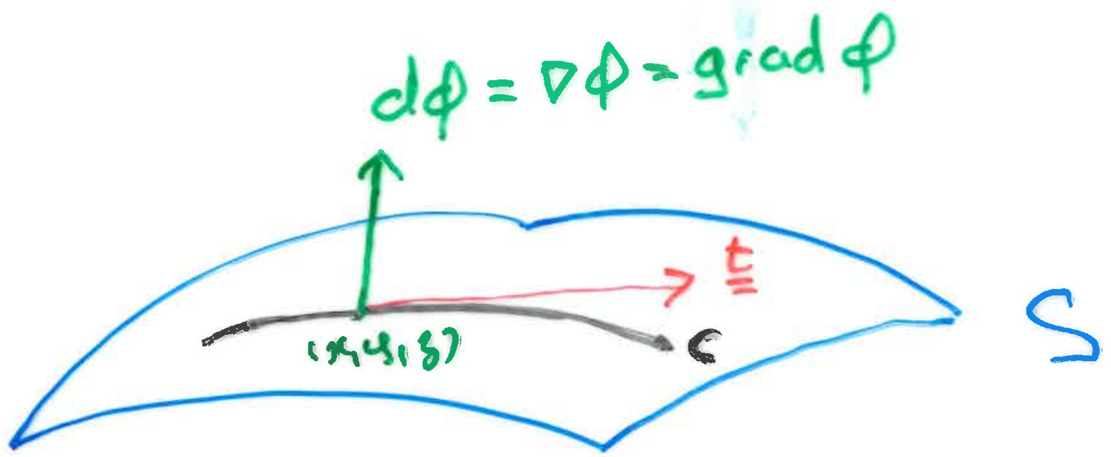
The equation

$$x^2+y^2+z^2 = 9$$

describes a sphere of radius 3, centred at the origin.

Let C be a curve on the surface S, parametrized as

$$C : \mathbb{R} \to S, \; t \mapsto (x(t), y(t), z(t))$$



Note that

$$\phi(x(t), y(t), z(t)) = k$$

with k the constant, <u>whatever</u> the curve C.

The chain rule give:

$$0 = \frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}$$

$$0 = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

Now

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \frac{dx}{dt}\underline{\underline{i}} + \frac{dy}{dt}\underline{\underline{j}} + \frac{dz}{dt}\underline{\underline{k}}$$

is a tangent $\underline{\underline{t}}$ to the curve C in the surface S.

Thus

$$\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) = \text{grad}\ \phi = d\phi$$

is a vector, depending on x, y, z, which is perpendicular to the surface S at (x, y, z).