Notation: For a function $F(x,y,z)$ we write

$$F_x = \frac{\partial}{\partial x} F, \qquad F_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} F\right)$$

Proposition: Let $\omega = F(x,y,z)$ be a differential 0-form. Suppose

$$F_{xy} = F_{yx}, \quad F_{xz} = F_{zx}, \quad F_{yz} = F_{zy}.$$

Then

$$d(d\omega) = 0.$$

Proof

$$d(d\omega) = d(F_x\,dx + F_y\,dy + F_z\,dz)$$

$$= d(F_x\,dx) + d(F_y\,dy) + d(F_z\,dz)$$

$$= (F_{xx}\,dx + F_{xy}\,dy + F_{xz}\,dz) \wedge dx$$

$$+ (F_{yx}\,dx + F_{yy}\,dy + F_{yz}\,dz) \wedge dy$$

$$+ (F_{zx}\,dx + F_{zy}\,dy + F_{zz}\,dz) \wedge dz$$

$$= F_{xx}\,\underbrace{\cancel{dx\wedge dx}}_{=0} + F_{xy}\,\cancel{dy}\wedge dx + F_{xz}\,\cancel{dz}\wedge dx$$

$$+ F_{yx}\,\cancel{dx}\wedge dy + F_{yy}\,\underbrace{\cancel{dy\wedge dy}}_{=0} + F_{yz}\,\cancel{dz}\wedge dy$$

$$+ F_{zx}\,\cancel{dx}\wedge dz + F_{zy}\,\cancel{dy}\wedge dz + F_{zz}\,\underbrace{\cancel{dz\wedge dz}}_{=0}$$

$$= 0 .$$

<u>Example</u> Prove that the 1-form

$$\omega = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$$

does not arise as the total derivative of any 0-form v. That is $\omega = dv$.

<u>Sol^n</u> Let's calculate $d\omega$. If $d\omega \neq 0$ then we can't have $d(dv) = 0$.

$$d\omega = (\cancel{6x}\,dx - 6z\,dy - 6y\,dz) \wedge dx$$

$$+ (3z\,dx + \cancel{2dy} + 3x\,dz) \wedge dy$$

$$+ (-4yz^2\,dx - 4xz^2\,dy - \cancel{8xyz}\,dz) \wedge dz$$

$$= 6z\,dx \wedge dy - 6y\,dz \wedge dx + 3z\,dx \wedge dy$$

$$- 3x\,dy \wedge dz + 4yz^2\,dz \wedge dx$$

$$- 4xz^2\,dy \wedge dz$$

$$= (6z + 3z)\,dx \wedge dy$$

$$+ (-3x - 4xz^2)\,dy \wedge dz$$

$$+ (-6y + 4yz^2)\,dz \wedge dx$$

$$\neq 0.$$

Differentiation of k-forms

A 2-form is an expression like

$$\omega = A\,dx \wedge dy + B\,dy \wedge dz + C\,dz \wedge dx$$

where A, B, C are functions of x, y, z.

A 3-form is an expression such as

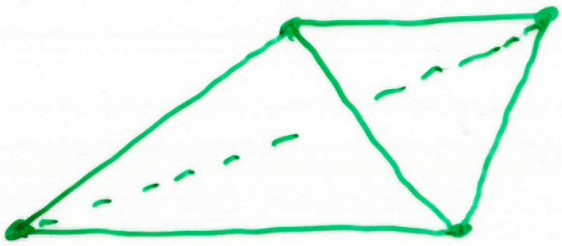
$$\omega = A\,dx \wedge dy \wedge dz + B\,dx \wedge dy \wedge dt + \cdots$$

where $A, B \cdots$ are functions of $x, y, z, t, \cdots$.

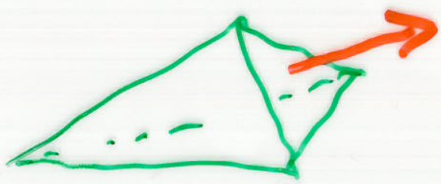
To understand integrals of 2-forms we needed to understand how to integrate constant 2-forms over oriented planar triangles.

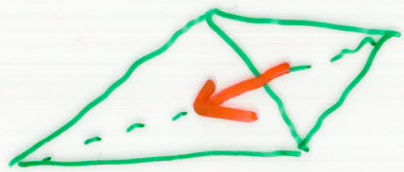
(areas)

For integrals of 3-forms we just need to understand how to integrate constant 3-forms over solid tetrahedra.



An orientation of such a tetrahedron can be specified by an arrow on its surface pointing either inwards or outwards.





Given a 2-form ω we can define a 3-form $d\omega$ such that

$$\int_{\partial S} \omega = \int_S d\omega$$

where S is an oriented 3-dimensional region.

The <u>derivative</u> $d\omega$ of a 2-form satisfies rules 1-6 of the last lecture, and also

7. $(dx \wedge dy) \wedge dz = dx \wedge (dy \wedge dz)$.

we usually write

$dx \wedge dy \wedge dz$

Exercise: Calculate $d\omega$ for

$$\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy.$$

<u>Sol</u>

Solⁿ

$$d\omega = dx \wedge dy \wedge dz + dy \wedge dz \wedge dx + dz \wedge dx \wedge dy$$

$$= dx \wedge dy \wedge dz + dx \wedge dy \wedge dz + dx \wedge dy \wedge dz$$

$$= 3\,dx \wedge dy \wedge dz.$$