$$\int_S \sin(x)$$ S 0-dimensional

$$\int_S \sin(x)\,dx$$ S 1-dimensional

$$\int_S \sin(x)\,dx \wedge dy$$ S 2-dimensional

$$\left(\sin(x) + \cos(x)\right) dx$$

$$= \sin(x)\,dx + \cos(x)\,dx$$

Example Find $d\omega$ where

$$\omega = A\,dx + B\,dy$$

with A, B functions of x, y.

Sol^n

$$d\omega = d(A\,dx + B\,dy)$$

$$= d(A\,dx) + d(B\,dy)$$

$$= (dA) \wedge dx + (dB) \wedge dy$$

$$= \left(\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy\right) \wedge dx$$

$$+ \left(\frac{\partial B}{\partial x}dx + \frac{\partial B}{\partial y}dy\right) \wedge dy$$

$$= \frac{\partial A}{\partial x}\underbrace{\cancel{dx \wedge dx}}_{=0} + \frac{\partial A}{\partial y}dy \wedge dx$$

$$+ \frac{\partial B}{\partial x}dx \wedge dy + \frac{\partial B}{\partial y}\underbrace{\cancel{dy \wedge dy}}_{=0}$$

$$= \frac{\partial A}{\partial y} dy \wedge dx + \frac{\partial B}{\partial x} dx \wedge dy$$

$$= -\frac{\partial A}{\partial y} dx \wedge dy + \frac{\partial B}{\partial x} dx \wedge dy$$

$$= \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx \wedge dy .$$

We've seen that rules 1-6 from last lecture are sufficient to derive the formula:

for $\omega = A\,dx + B\,dy$

we get

$$\boxed{d\omega = \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx \wedge dy .} \quad (*)$$

To motivate rules 1-6 we'll explain why (*) is precisely what is needed for Stoke's formula to hold.

So suppose

$$\omega = A\,dx + B\,dy$$

where A, B are functions of x and y.

We want to define

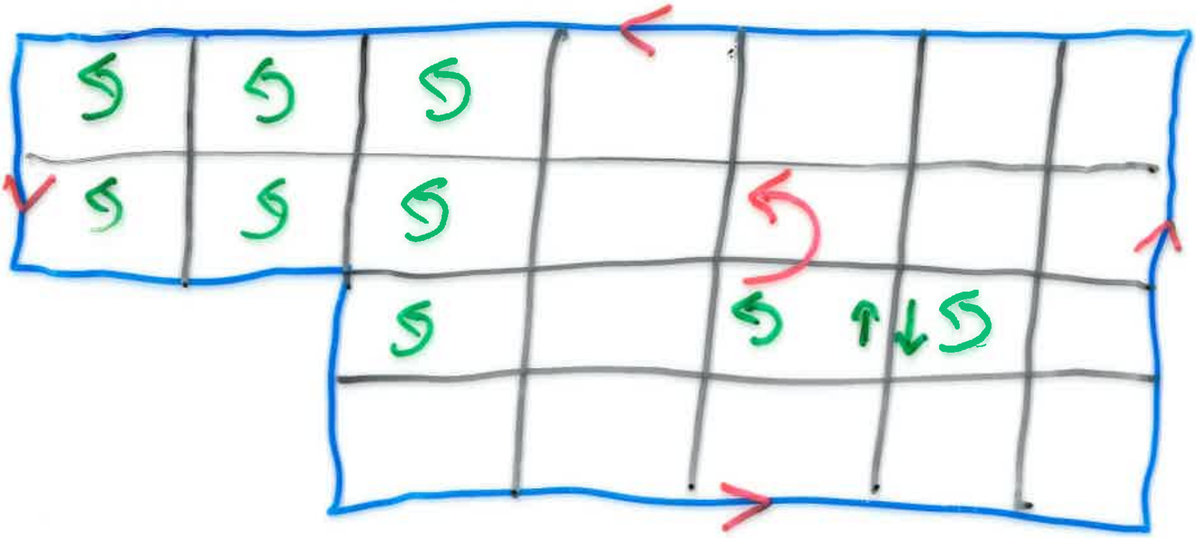
$$d\omega = C\,dx \wedge dy$$

where C is a function of x, y, such that

$$\boxed{\int_{\partial S} A\,dx + B\,dy = \int_S C\,dx \wedge dy}$$

What does C have to be?

For simplicity let's suppose that S is an oriented region in the xy-plane, with boundary ∂S oriented accordingly.



S

$$S = S_1 \cup S_2 \cup \cdots \cup S_n$$

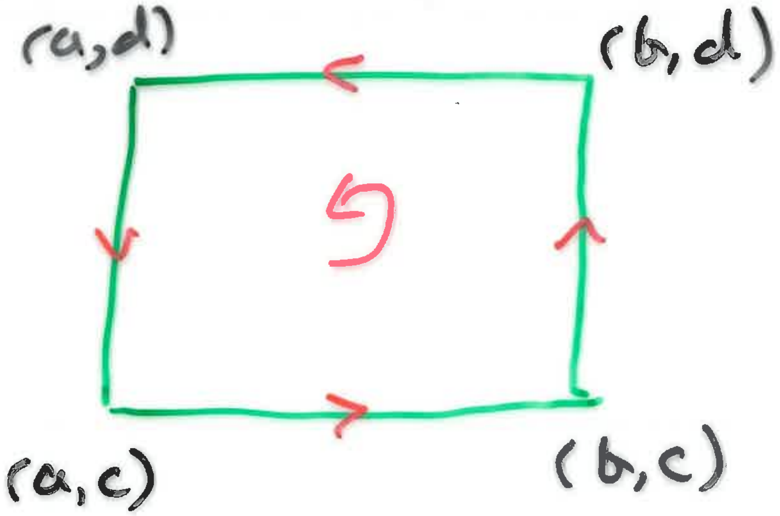
Note:

$$\int_{\partial S} A\,dx + B\,dy = \sum_{i=1}^{n} \int_{\partial S_i} A\,dx + B\,dy$$

Thus, whatever the region S, we only need to consider small squares S_i. For each S_i we need

$$\boxed{\int_{\partial S_i} A\,dx + B\,dy = \int_{S_i} C\,dx \wedge dy}$$

Suppose S_i is the square



$a \le x \le b, \quad c \le y \le d.$

We have

$$\int_{\partial S_i} A\,dx + B\,dy$$

$$= \int_a^b A(x,c)\,dx + \int_c^d B(b,y)\,dy$$

$$+ \int_b^a A(x,d)\,dx + \int_d^c B(a,y)\,dy$$

$$= \int_c^d \left(B(b,y) - B(a,y) \right) dy$$

$$- \int_a^b \left(A(x,d) - A(x,c) \right) dx$$

$$\underset{FTC}{=} \int_c^d \left(\int_a^b \frac{\partial B}{\partial x}\, dx \right) dy$$

$$- \int_a^b \left(\int_c^d \frac{\partial A}{\partial y}\, dy \right) dx$$

$$= \int_{S_i} \frac{\partial B}{\partial x}\, dx \wedge dy$$

$$- \int_{S_i} \frac{\partial A}{\partial y}\, dx \wedge dy$$

$$= \int_{S_i} \underbrace{\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right)}_{c} dx \wedge dy$$

Thus, we need

$$d\omega = c\, dx \wedge dy = \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx \wedge dy$$